Lesson 17

Continuation of Section 14.7

Find local maximum, minimum and saddle points of $f(x,y)$

Step 1: Find the critical points of $f(x,y)$.

$$f_x(x,y) = 0, \quad f_y(x,y) = 0$$

Step 2: Find f_{xx}, f_{xy} and f_{yy} and evaluate them at the critical points

Step 3: $D = f_{xx} f_{yy} - (f_{xy})^2$.
at the critical points.

Step 4

If $D < 0$, the critical point is a saddle point.

If $\underline{D > 0}$.

A) $f_{xx} > 0$, the critical point is a local minimum

B) $f_{xx} < 0$, the critical point is a local maximum.

If $\underline{D = 0}$ the test does not apply.

Example: Find local max, min and saddle points of

$$f(x,y) = x^3 + y^3 - 3xy.$$

Step I: $f_x(x,y)=0$, $f_y(x,y)=0$

$$\boxed{f_x(x,y) = 3x^2 - 3y, \quad f_y(x,y) = 3y^2 - 3x}$$

$3x^2 - 3y = 0$, $\quad y = x^2$.

$3y^2 - 3x = 0$, $\quad x = y^2$

$y = \underline{x}^2$ and $x = \underline{y}^2$

$y = (y^2)^2$; $\quad y = y^4$.

$y - y^4 = 0 \quad y(1-y^3) = 0$

Either $y=0$ or $y^3=1$

$y=1$

$y=0$ or $y=1$.

when $\underline{y=0}$, $x=y^2$, $x=0$

$(0,0)$.

when $y=1$, $x=y^2$, $x=1$

$(1,1)$

<u>Critical points</u>: $(0,0)$; $(1,1)$.

<u>Step II</u>: $f_{xx}=6x$; $f_{yx}=-3$.

$f_{yy}=6y$.

$D=f_{xx}f_{yy}-(f_{xy})^2=\underline{36xy-9}$

when $x = y = 0$

$\boxed{D = -9}$ Saddle point.

when $x = y = 1$

$D = 36 - 9 = 27. > 0$

Check the sign of f_{xx}

$f_{xx} = 6x$, when $x = 1$

$f_{xx} = 6 > 0$

Local minimum.

Question: when $D > 0$

$$D = \underbrace{f_{xx} f_{yy}}_{>0} \underbrace{- (f_{xy})^2}_{<0} > 0$$

@ ~~[illegible]~~

If $D > 0$, then

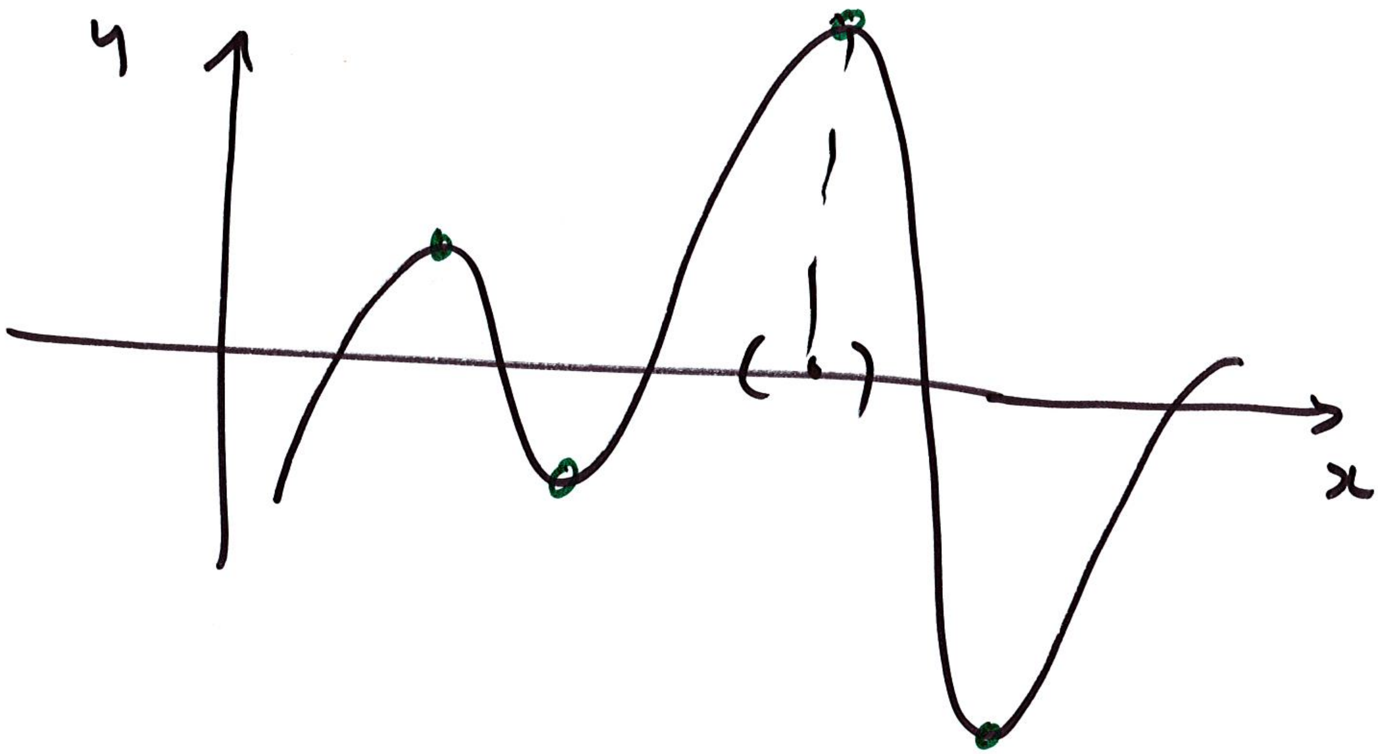
$$f_{xx} f_{yy} > 0.$$

this means that f_{xx} and f_{yy} have the same sign.

and so this must
be a critical point.

Find all critical points
of $f(x,y)$ evaluate
the function at these
points and see
which one is the max
and which is the min.

(Absolute)
Global maximum and minimum.



What is the largest value of f?

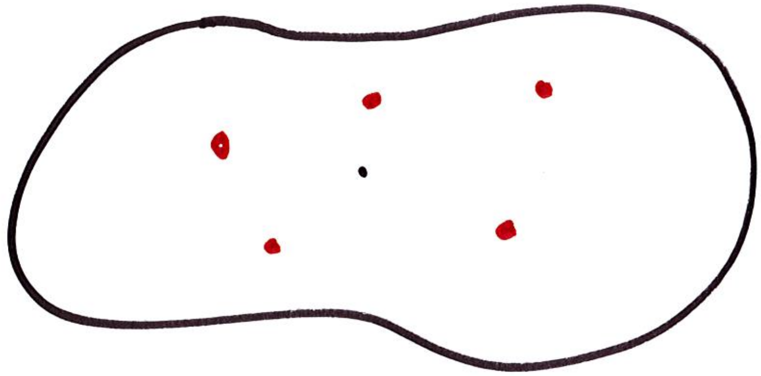
What is the smallest value of f?

If $f(a,b)$ is the largest value of f on $(-\infty, \infty)$

Then $f(a,b)$ is also local max.

If $f(x,y)$ is defined

on a region



D

To find max and min

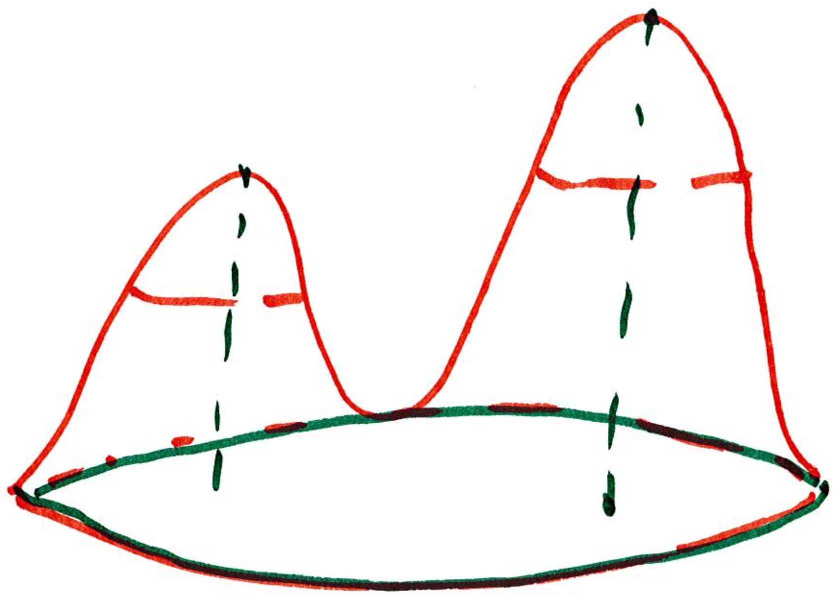
of $f(x,y)$ on D

we look for critical points

inside D.

If the max or min occurs

inside



Then we find a way to look for the max and min at the boundary of D.

Recall the case of

$f(x)$ on an interval

$[a, b]$.

Example: Find the max

and min of

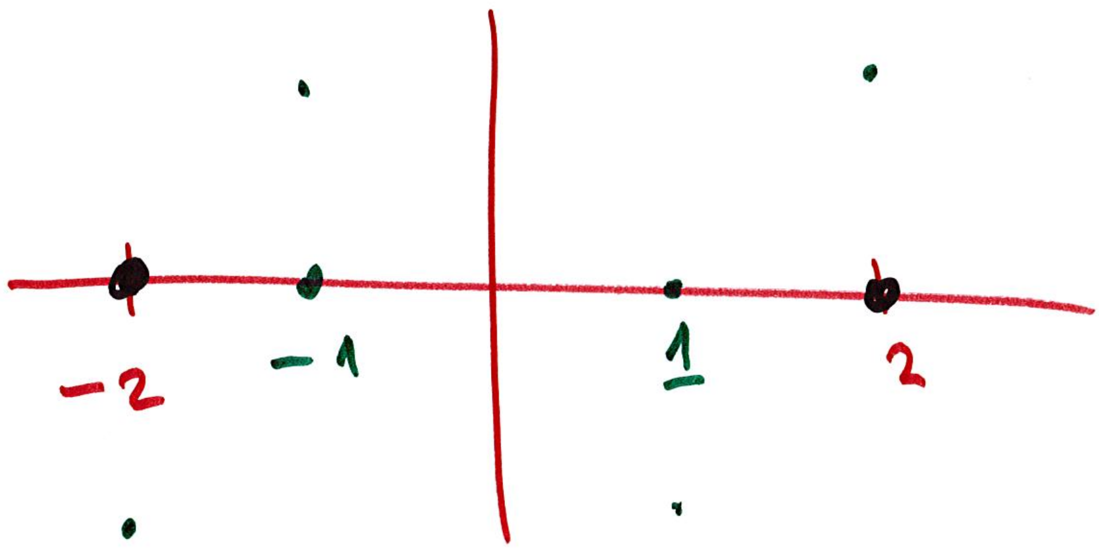
$$f(x) = x^3 - 3x$$

on $[-2, 2]$.

Search for critical points inside

$$f'(x) = 3x^2 - 3 = 0$$

$x^2 - 1$ $\quad X = 1$ or $X = -1$



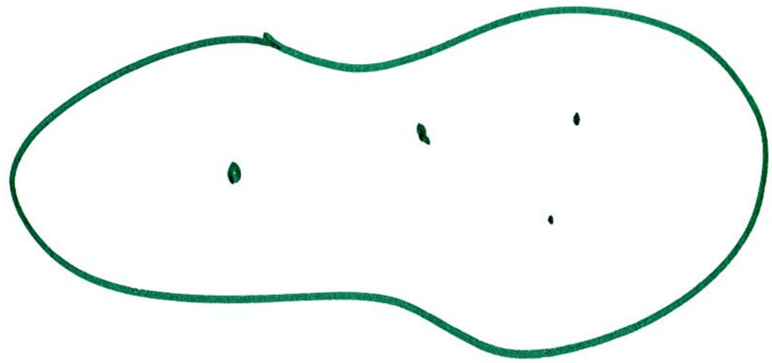
$$f(-1) = -1+3 = 2$$

$$f(1) = 1-3 = -2.$$

Check the end points.

$$f(2) = 8-6 = 2$$

$$f(-2) = -8+6 = -2.$$

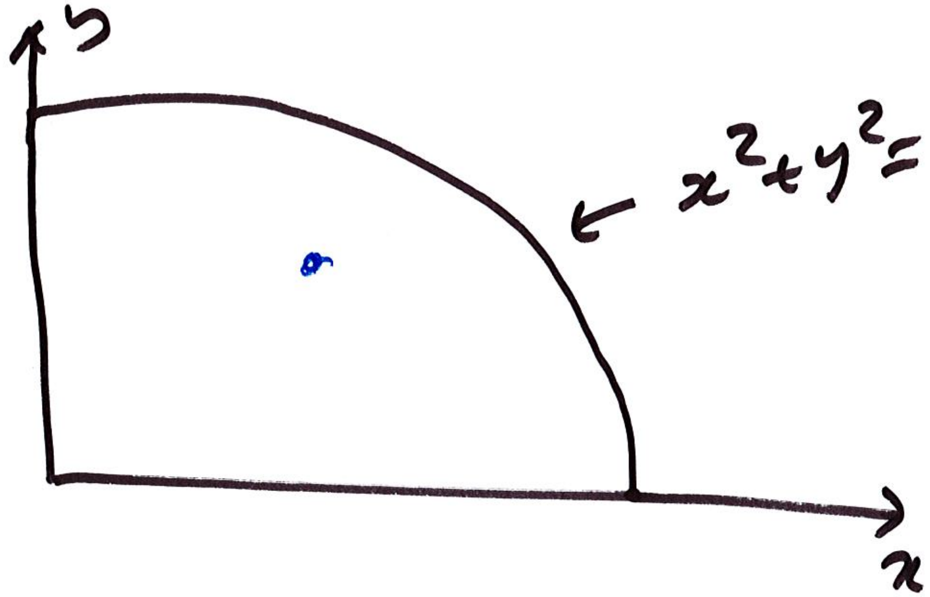


$f(x,y)$

<u>Example:</u> $f(x,y) = xy^2 + 5$

Find the absolute max and min of f on the region

$$D = \{ x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq 3 \}$$



$$f(x,y) = xy^2 + 5.$$

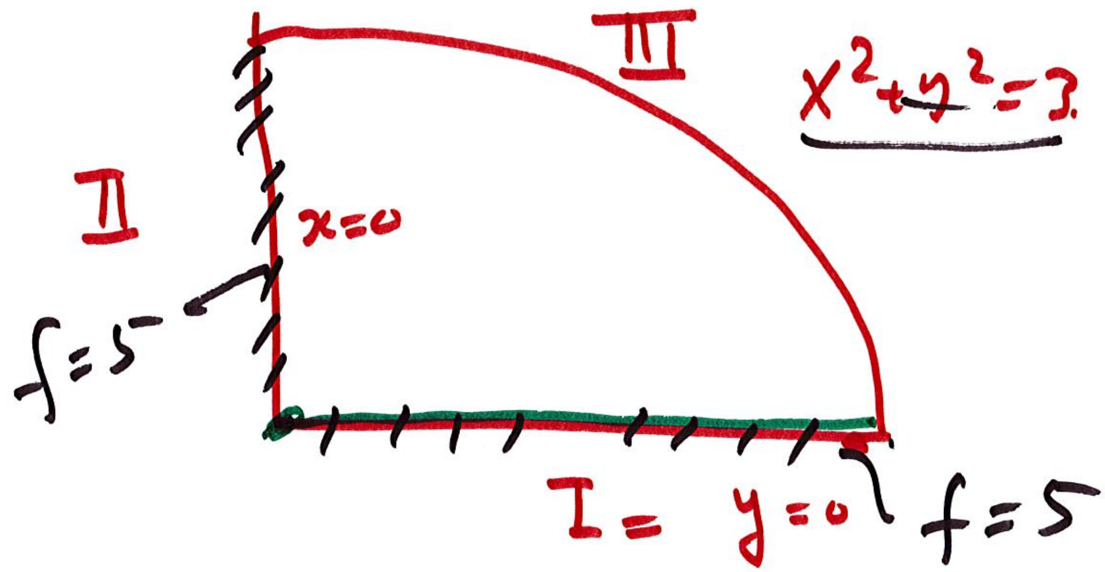
Step 1: Search for critical points ~~ines~~ inside the region.

$$f_x = y^2; \quad f_y = 2xy$$

$$f_x = 0 \text{ if } y = 0, \quad f_y = 0 = 2xy$$

$$x = 0 \text{ or } y = 0$$

Step 2: Find max and min on the boundary of D.



$$f(x,y) = xy^2 + 5$$

$$f(0,y) = 5 \quad , \quad f(x,0) = 5 .$$

$$\text{on } x^2+y^2 = 3, \quad y^2 = 3-x^2$$

$$f(x,y) = x(3-x^2)+5 . \text{ on } \overline{\underline{\mathrm{III}}}$$

$$g(x) = x(3-x^2)+5$$

$$0 \le x \le \sqrt{3} .$$

Now we have to find max and min of

$$g(x) = x(3-x^2) + \underline{\underline{5}}$$

for x in $[0, \sqrt{3}]$.

$$g(x) = 3x - x^3 + 5$$

$$g'(x) = 3 - 3x^2 = 0$$

$$3x^2 = 3, \quad x^2 = 1$$

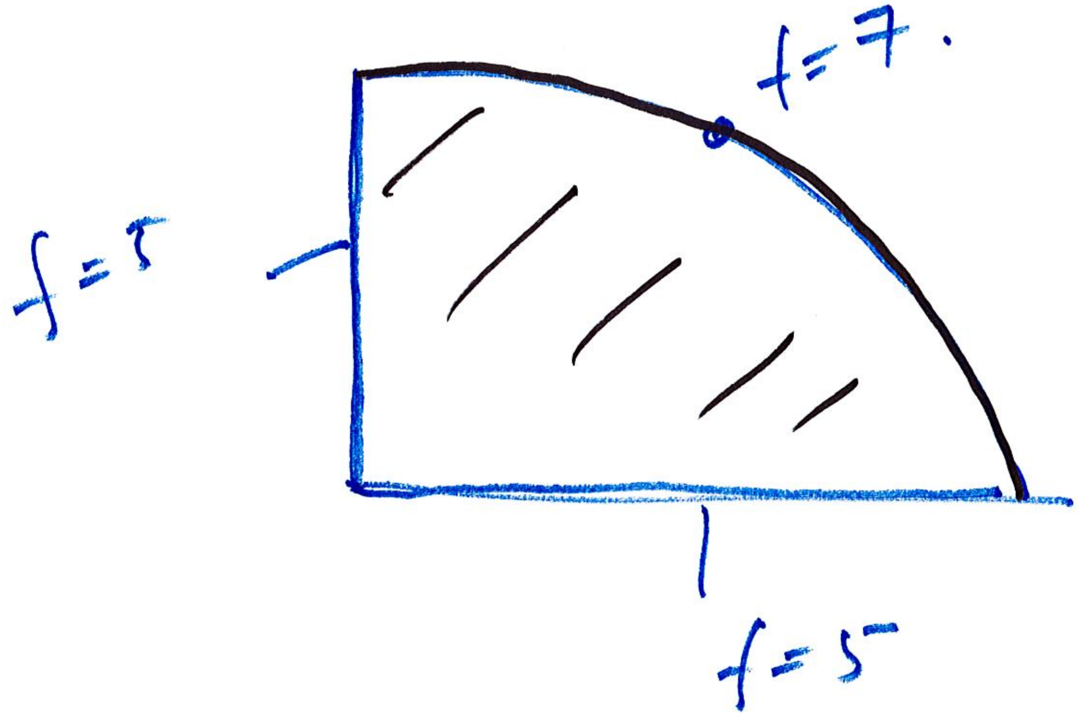
$x = 1$ or $\boxed{x = -1}$ outside $[0, \sqrt{3}]$

We evaluate $\boxed{g(1) = 3 - 1 + 5 = 7}$

$g(0) = 5$, $g(\sqrt{3}) = 5$.

Conclusion:

$$f(x,y) = \underline{xy^2} + 5.$$



Absolute max = 7

Absolute min = 5