Lesson 20 Section 15.2

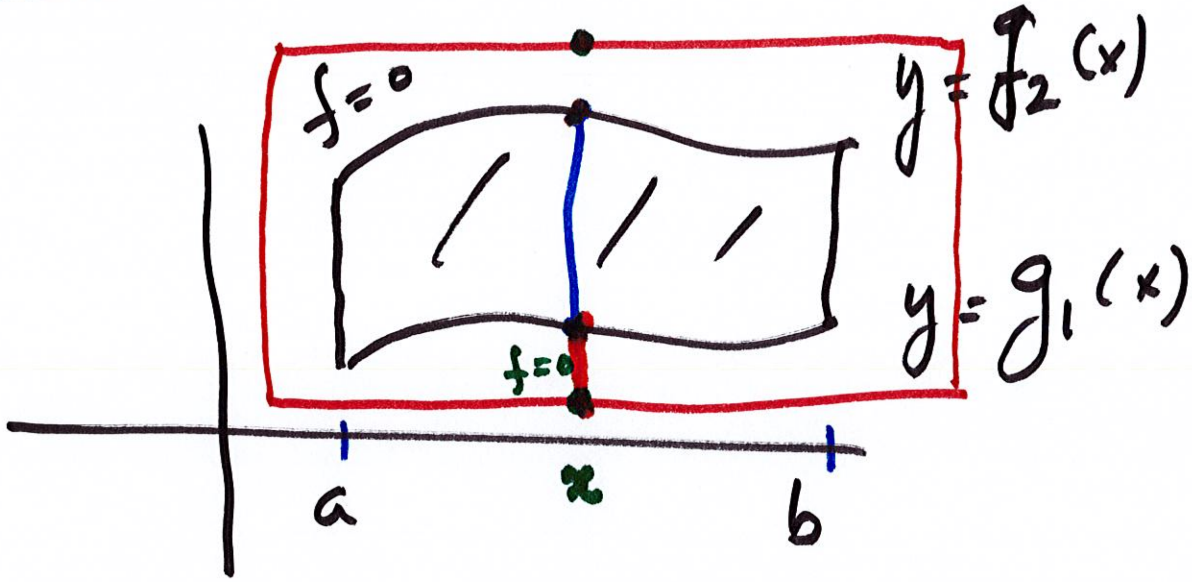
Double Integrals In General Regions

$f(x,y)$ continuous on D

Integrate $\iint_D f(x,y)\,dA$.

Last time we did the case where D was a rectangle.

Regions of Type I



$$D = \{ (x,y) : \ g_1(x) \le y \le g_2(x)$$

$$a \le x \le b \ \}$$

$$\iint_D f(x,y)\, dA =$$

<u>Idea:</u> Take a rectangle R which contains D.

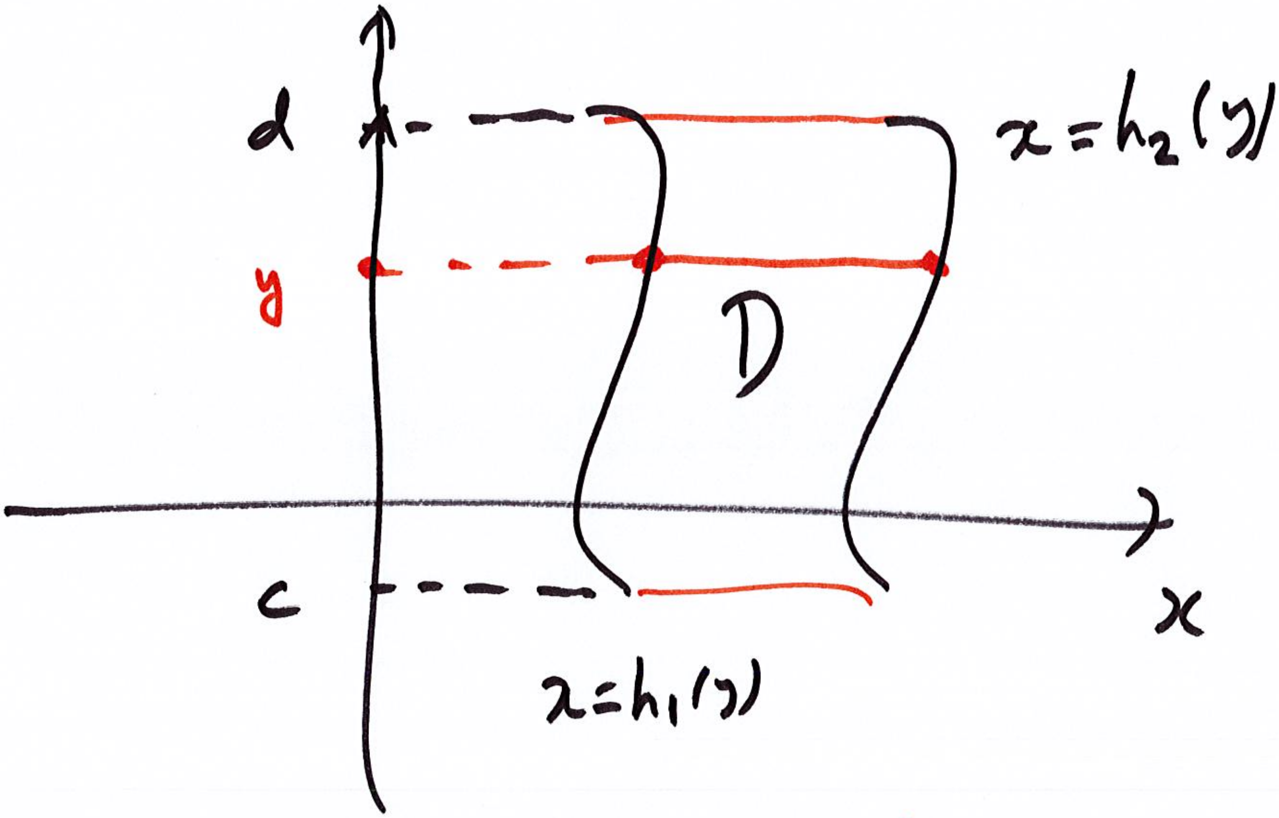
and think of $f(x,y)=0$

for (x,y) outside D.

$$\iint_D f(x,y)\,dA = \iint_R f(x,y)\,dA$$

$$= \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx$$

Regions of Type II



Switch the roles of x and y.

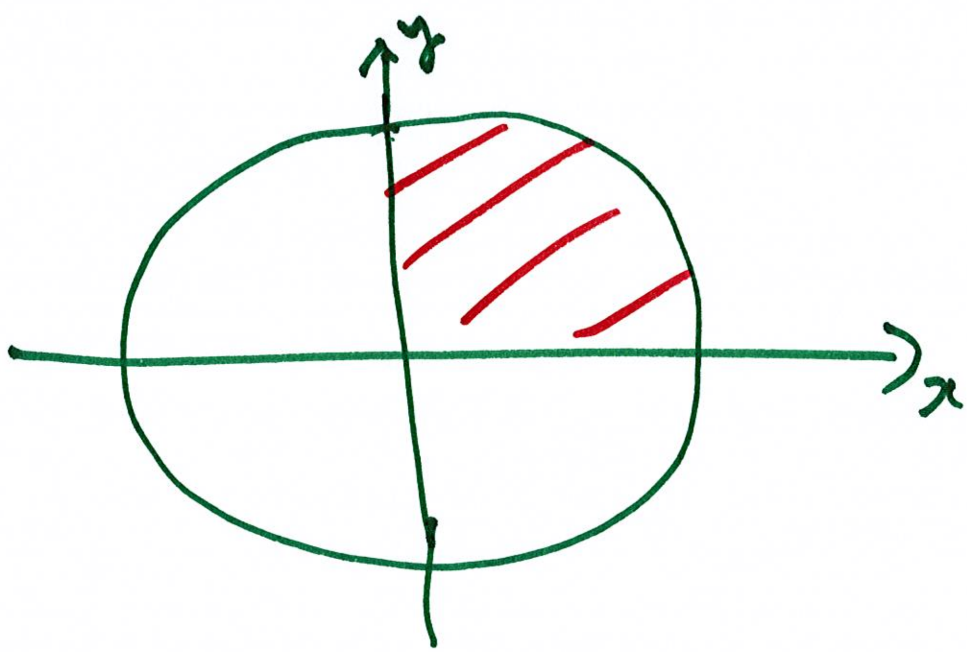
$$\iint_D f(x,y)\, dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\, dx \right] dy$$

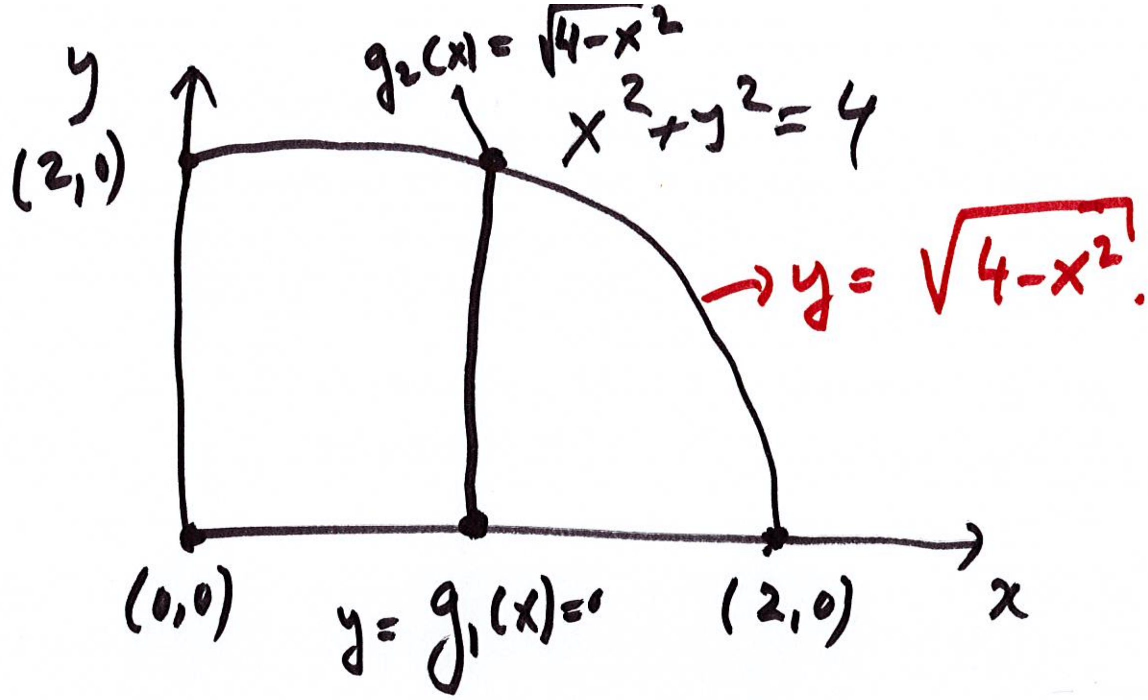
Examples:

1) Compute $\displaystyle\iint_D xy \, dA$

$$D = \{ \underbrace{x^2+y^2 \le 4}, \ x \geq 0, \ y \geq 0 \}$$

Step 1: Sketch the region.





$$\iint_D xy\, dA = \int_0^2 \left[\int_0^{\sqrt{4-x^2}} x \underbrace{y\, dy} \right] dx$$

$$= \int_0^2 x \left(\int_0^{\sqrt{4-x^2}} y\, dy \right) dx$$

$$= \int_0^2 x \, \frac{1}{2}(4-x^2)\, dx .$$

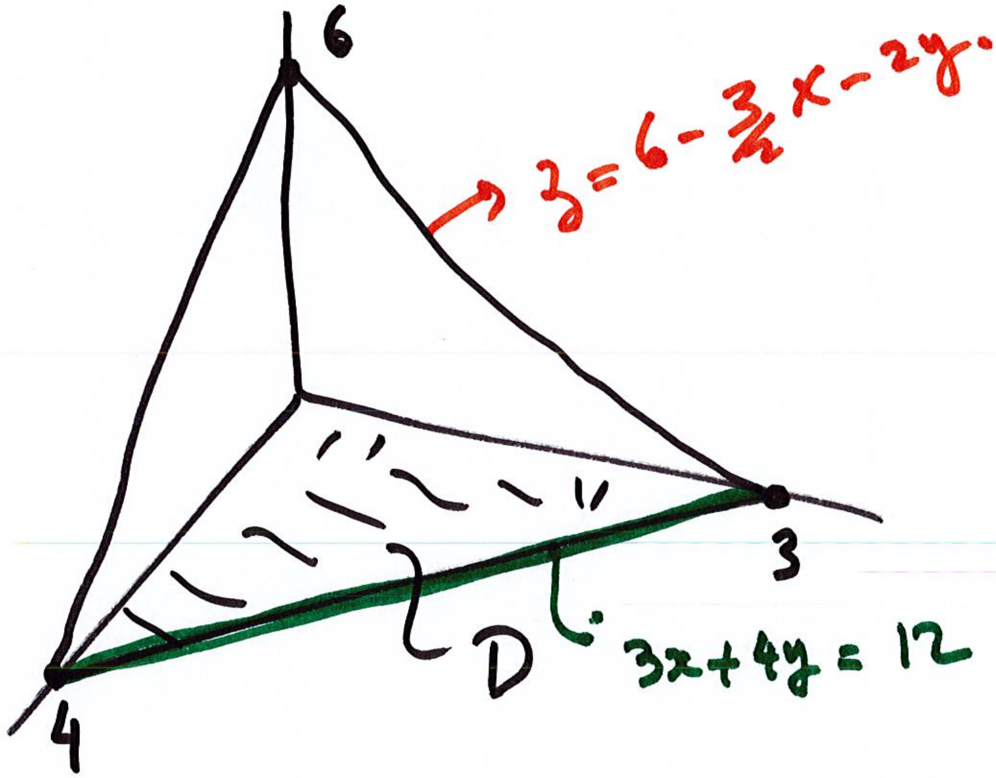
$$\int_0^{\sqrt{4-x^2}} y\, dy = \frac{1}{2} y^2 \Big|_0^{\sqrt{4-x^2}} = \frac{1}{2}(4-x^2).$$

$$= \frac{1}{2} \int_0^2 (4x - x3 \ ) \, dx$$

$$= \frac{1}{2} \left[2x^2 - \frac{x^4}{4} \right]_0^2$$

$$= \frac{1}{2} \left[8 - 4 \right] = 2.$$

2) Find the volume of the pyramid



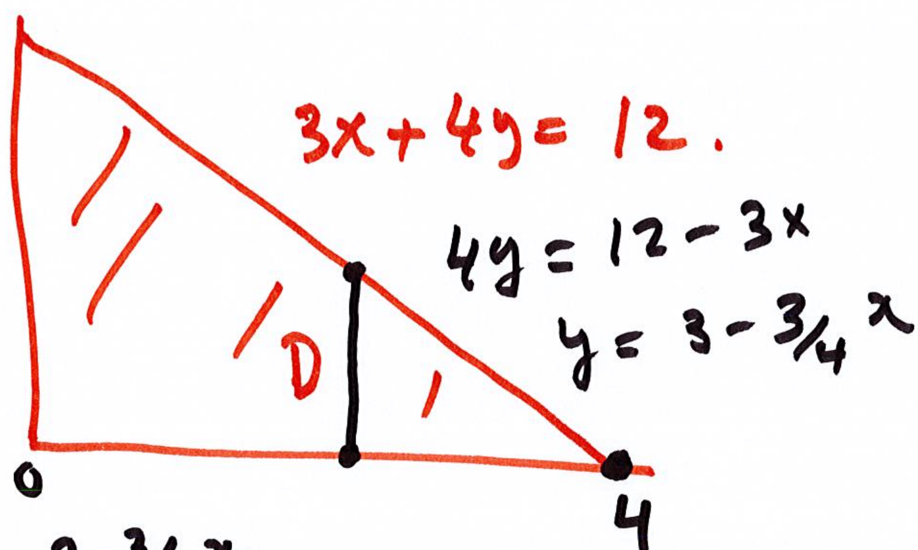
$$3x + 4y + 2z = 12.$$

$$2z = 12 - 3x - 4y$$

$$\boxed{z = 6 - \frac{3}{2}x - 2y.}$$

The Volume of the pyramid =

$$\iint_D \left[6 - \tfrac{3}{2}x - 2y\right] dA$$



$$= \int_0^4 \left[\underbrace{\int_0^{3 - \frac{3}{4}x} \left[6 - \tfrac{3}{2}x - 2y\right] dy}_{(3 - 3/4\, x)^2} \right] dx$$

$$\int_0^{3-3/4\,x} \Big[\underbrace{\left(6 - \tfrac{3}{2}x\right)} - 2y \Big]\, \underline{dy}.$$

$$= \overbrace{\left(6 - \tfrac{3}{2}x\right)}^{=2\left(3-\frac{3}{4}x\right)}\left(3 - \tfrac{3}{4}x\right) - \left(3 - \tfrac{3}{4}x\right)^2$$

$$= 2\left(3 - \tfrac{3}{4}x\right)\left(3 - \tfrac{3}{4}x\right) - \left(3 - \tfrac{3}{4}x\right)^2$$

$$= 2\left(3 - \tfrac{3}{4}x\right)^2 - \left(3 - \tfrac{3}{4}x\right)^2$$

$$= \left(3 - \tfrac{3}{4}x\right)^2.$$

$$\iint_D \left(6 - \tfrac{3}{4}x - 2y\right) dA = \int_0^4 \left(3 - \tfrac{3}{4}x\right)^2 dx.$$

Set $u = 3 - 3/4\,x$.

when $x = 0$, $u = 3$

$x = 4$, $u = 0$

$du = -3/4\, dx$

$dx = -4/3\, du$

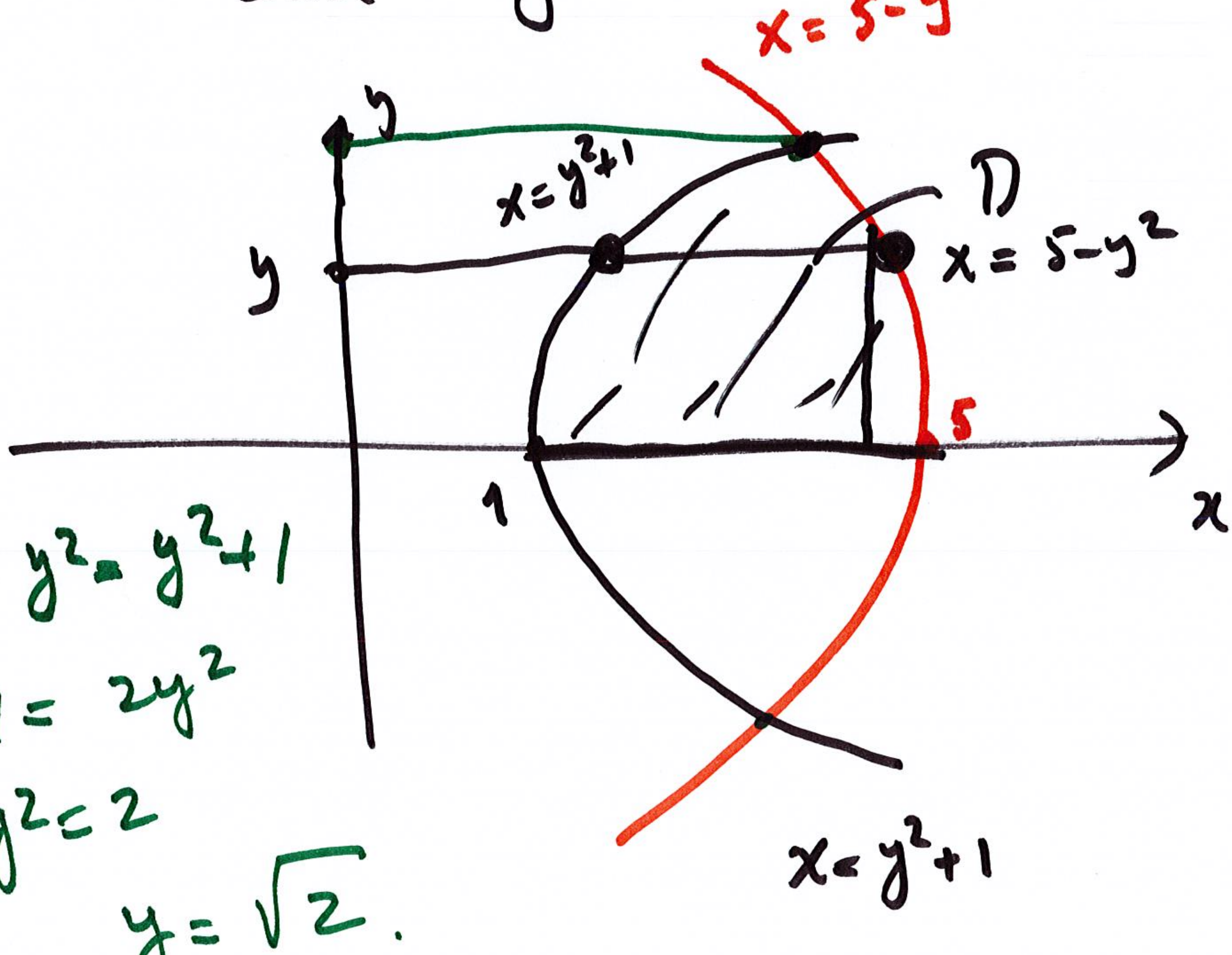
$$\int_0^4 \left(3 - 3/4\, x\right)^2 dx = \int_3^0 u^2 \left(-4/3\, du\right)$$

$$= 4/3 \int_0^3 u^2\, du = \frac{4}{3} \cdot \frac{u^3}{3} \Big|_0^3$$

$$= \frac{4}{3} \cdot 9 = 12.$$

Compute $\iint_D x\,y\,dA$

D is the region bounded by $x = y^2+1$, $x = 5-y^2$ and $y \geq 0$.



$5-y^2 = y^2+1$

$4 = 2y^2$

$y^2 = 2$

$y = \sqrt{2}$.

$$\iint_D xy\,dA = \int_0^{\sqrt{2}} \left[\int_{y^2+1}^{5-y^2} x\underline{\underline{y}}\,dx \right] dy$$

$$= \int_0^{\sqrt{2}} y \left(\int_{y^2+1}^{5-y^2} x\,dx \right) dy$$

$$= \int_0^{\sqrt{2}} y \cdot \frac{1}{2} \left((5-y^2)^2 - (1+y^2)^2 \right) dy$$

$$= \frac{1}{2} \int_0^{\sqrt{2}} y \left(25 - 10y^2 + \cancel{y^4} - 1 - 2y^2 - \cancel{y^4} \right) dy$$

$$= \frac{1}{2} \int_0^{\sqrt{2}} y (24 - 12y^2)\,dy$$

$$= 12 \int_0^{\sqrt{2}} y \, dy - 6 \int_0^{\sqrt{2}} y^3 \, dy$$

$$= 12 \frac{(\cancel{2})}{\cancel{2}} - \frac{6}{4} \cdot (\sqrt{2})^4$$

$$= 12 - \frac{6}{4} \cdot 4 = 12 - 6 = 6$$