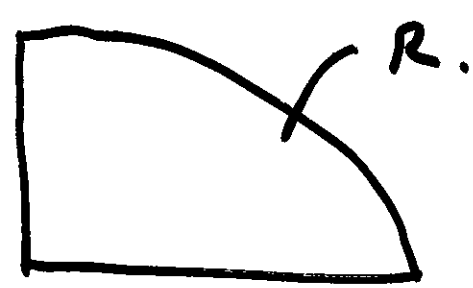
Lesson 21   Section 15.3

Double Integrals In Polar Coordinates.

Why we want to do this?

This is very useful in cases where one has circular symmetry.

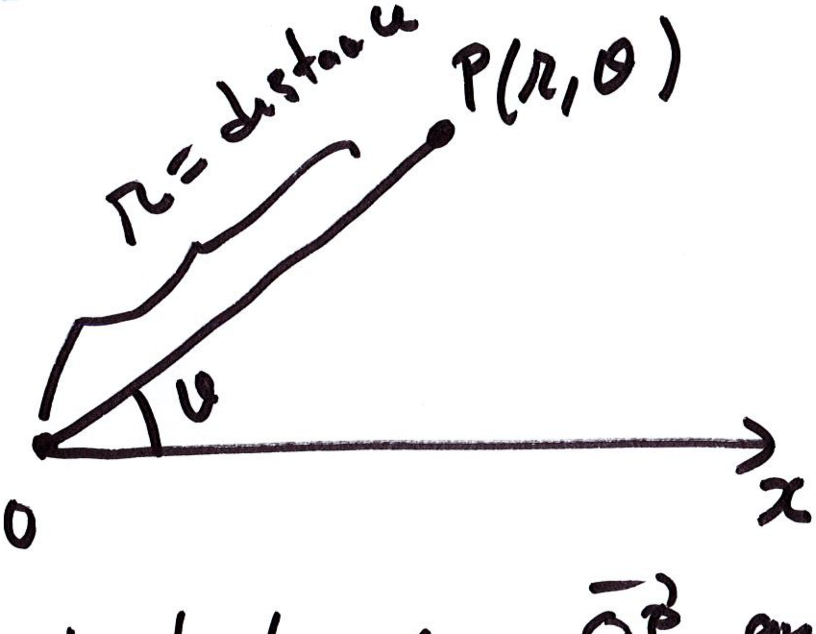
Ex: $$\iint_R e^{x^2+y^2}\, dA$$



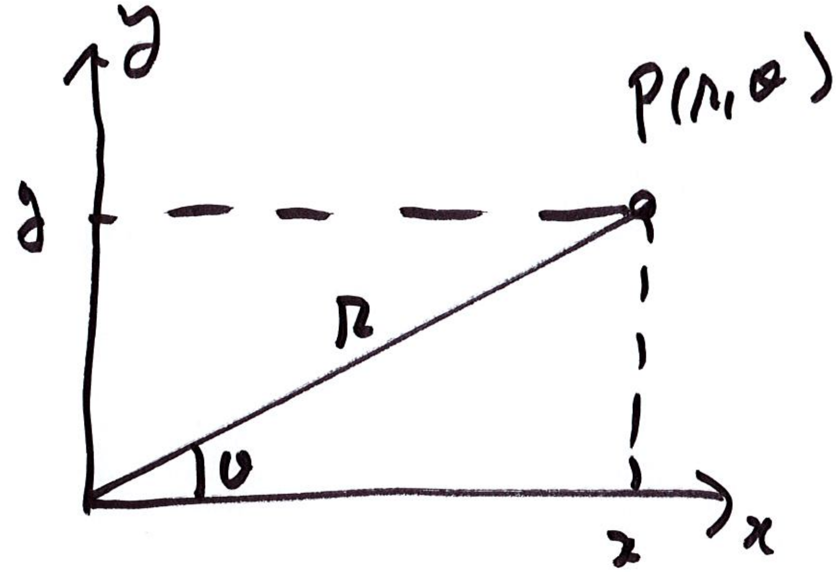
$$\int \left[\int e^{x^2+y^2}\,dy \right] dx$$

$$= \int e^{x^2} \left[\underbrace{\int e^{y^2}\,dy}_{?} \right] dx$$

Polar Rectangles



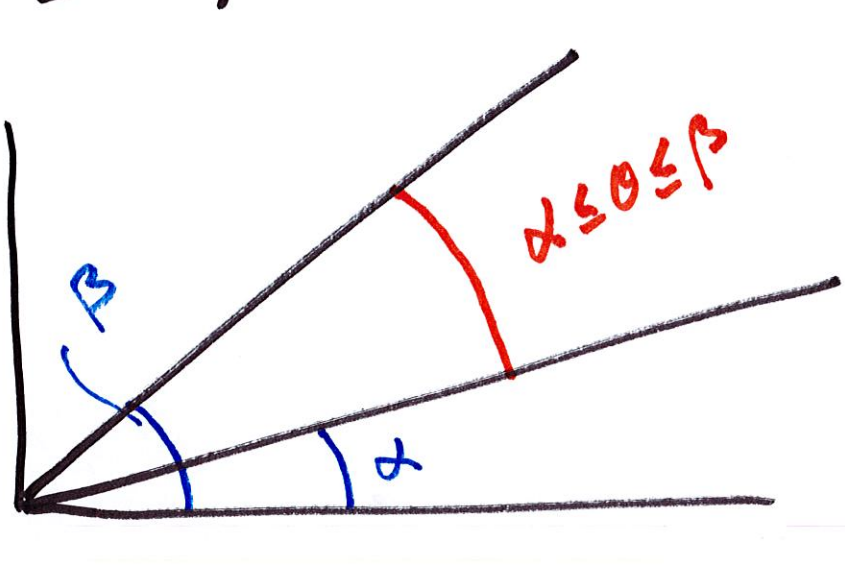
θ = angle between $\overrightarrow{OP}$ and the x-axis.



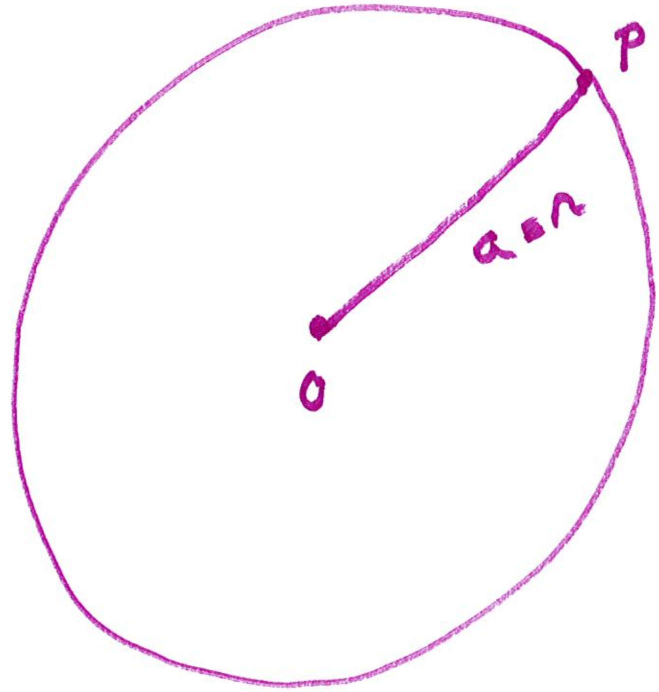
$x = r\cos\theta, \quad y = r\sin\theta.$

<u>Polar rectangle</u>: is a region

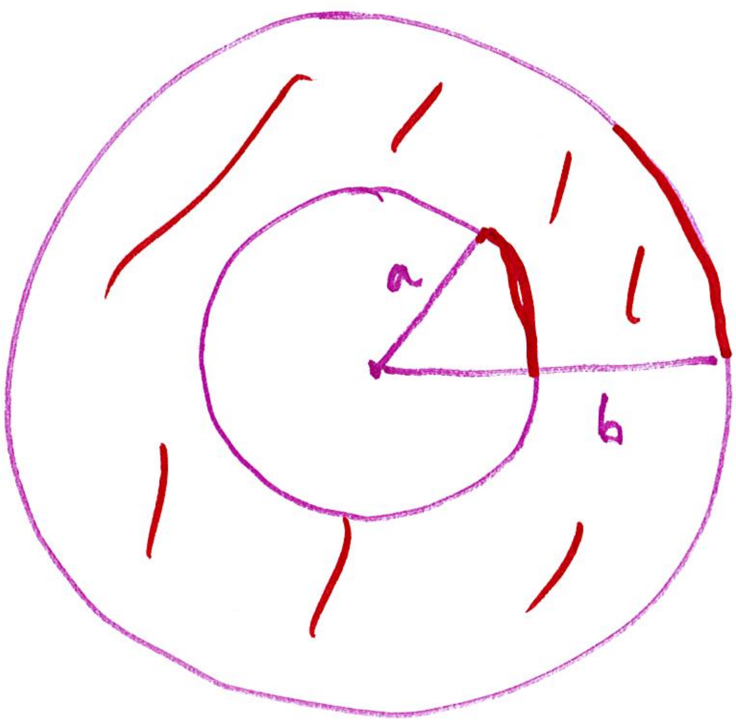
$a \le r \le b, \quad \alpha \le \theta \le \beta.$

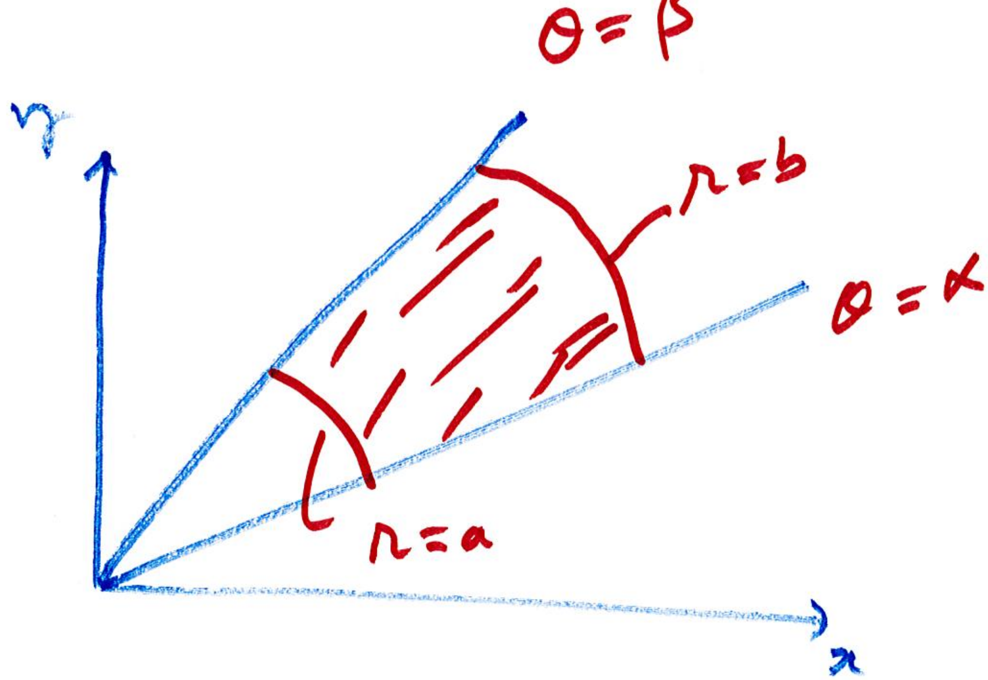


$r = a$



$a \le r \le b$





$$\iint_R f(x,y)\, dA$$

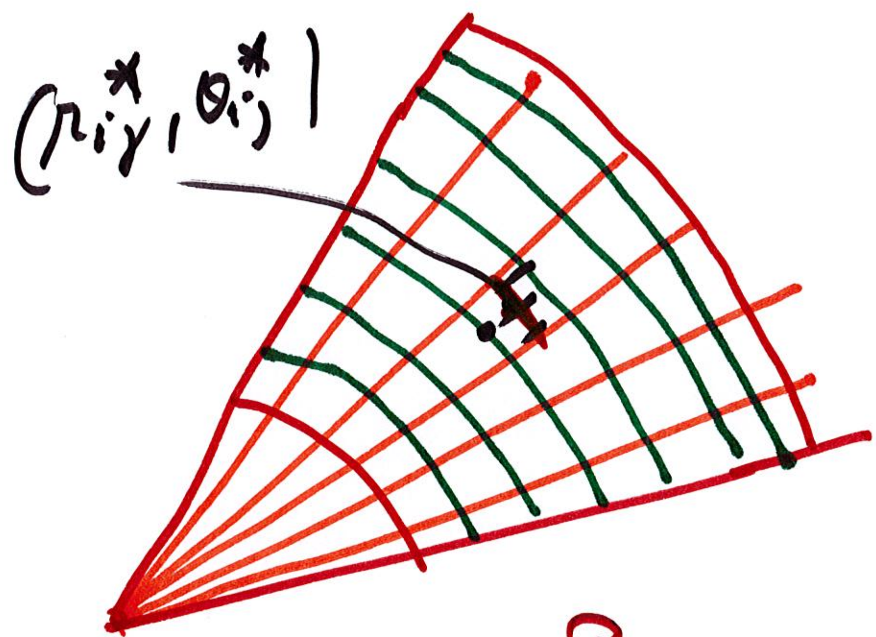
$R =$ Polar rectangle

$a \le r \le b$

$\alpha \le \theta \le \beta.$

<u>Step 1</u>: divide $[a,b]$ into m equal parts

divide $[\alpha, \beta]$ into n equal parts.



$x = r\cos\theta$

$y = r\sin\theta$

Form the sum

Pick r_{ij}^* = average of radii.

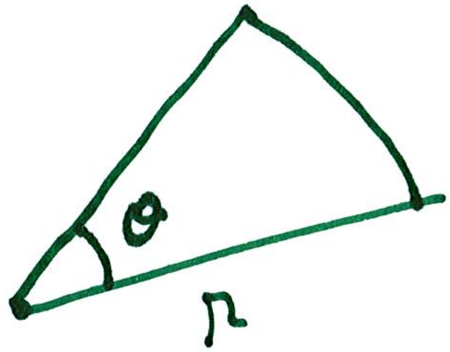
$$f(r_{ij}^*\cos\theta_{ij}^*,\ r_{ij}^*\sin\theta_{ij}^*)\,\underbrace{\Delta A}_{r_{ij}^*\,\Delta r\,\Delta\theta}$$

ΔA = area of the small polar rectangle.

We may think that $\Delta A = \Delta r\,\Delta\theta$

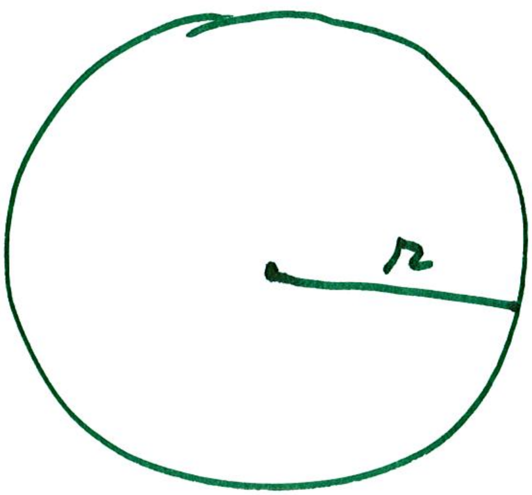
But that is not the case.

Fact:



$$\text{Area} = \frac{1}{2}\theta r^2$$

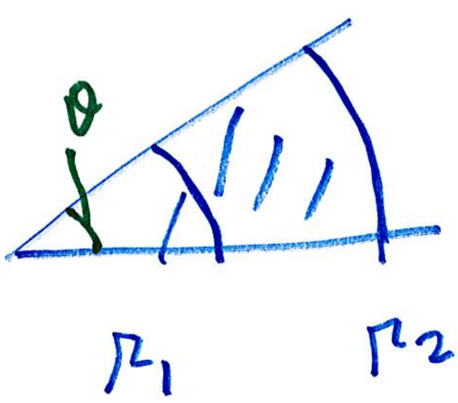
Example:



$\theta = 2\pi$

$$A = \frac{1}{2}\cdot 2\pi r^2$$

$$= \pi r^2.$$

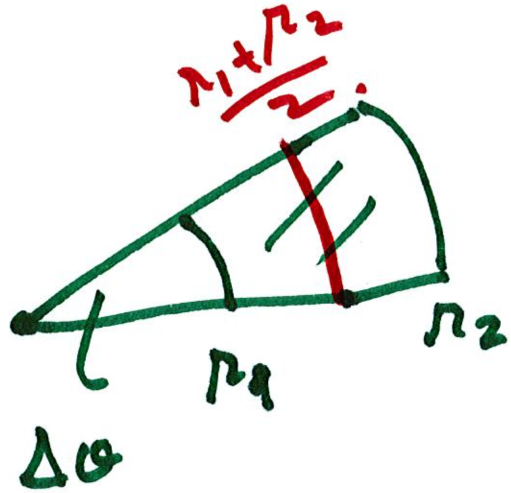


$$A = \frac{1}{2}\theta r_2^2 - \frac{1}{2}\theta r_1^2$$

$$= \frac{1}{2}\theta\left(r_2^2 - r_1^2\right)$$

$$= \frac{1}{2}\theta\left(r_2 + r_1\right)\left(r_2 - r_1\right)$$

$$r_2 - r_1 = \Delta r.$$



$$\boxed{\text{Area} = \frac{1}{2}(r_1+r_2).\ \Delta\theta.\ \Delta r.}$$

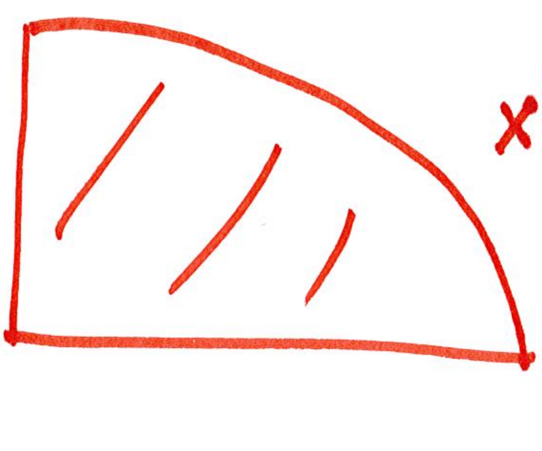
$$\lim_{\substack{m\to\infty \\ n\to\infty}} \sum_{\substack{1\le i\le m \\ 1\le j\le n}} f(r_{ij}^* \cos\theta_{ij}^*, r_{ij}^* \sin\theta_{ij}^*)\, r_{ij}^*\, \Delta\theta\, \Delta r$$

$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Conclusion:

$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

<u>Example:</u> $$\iint_R e^{x^2+y^2}\,dA$$

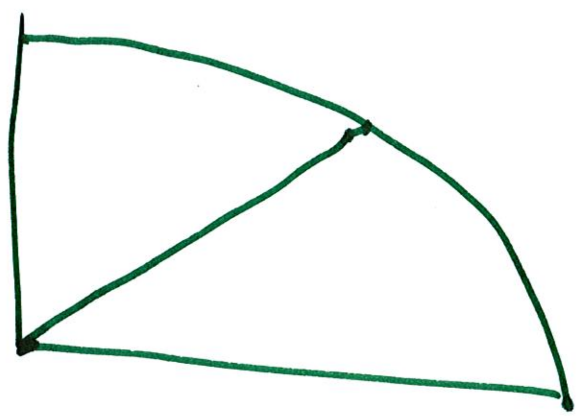


$x^2+y^2 \leq 1$

$x \geq 0$

$y \geq 0.$

Region in Polar Coordinates.



$0 \leq r \leq 1$

$0 \leq \theta \leq \pi/2$

$x = r\cos\theta, \quad y = r\sin\theta$

$$\iint_R e^{x^2+y^2}\,dA = \int_0^{\pi/2}\int_0^1 e^{r^2\cos^2\theta + r^2\sin^2\theta}\, r\,dr\,d\theta$$

$$r^2\cos^2\theta + r^2\sin^2\theta = r^2.$$

$$\int_0^{\pi/2}\Big[\underbrace{\int_0^1 e^{r^2}\,\underline{\underline{r}}\,dr}\Big]d\theta. =$$

$r^2 = u$, $du = 2r\,dr$.

$$\int_0^1 e^{r^2}\,\underbrace{r\,dr} = \int_0^1 e^u\,\frac{du}{2}.$$

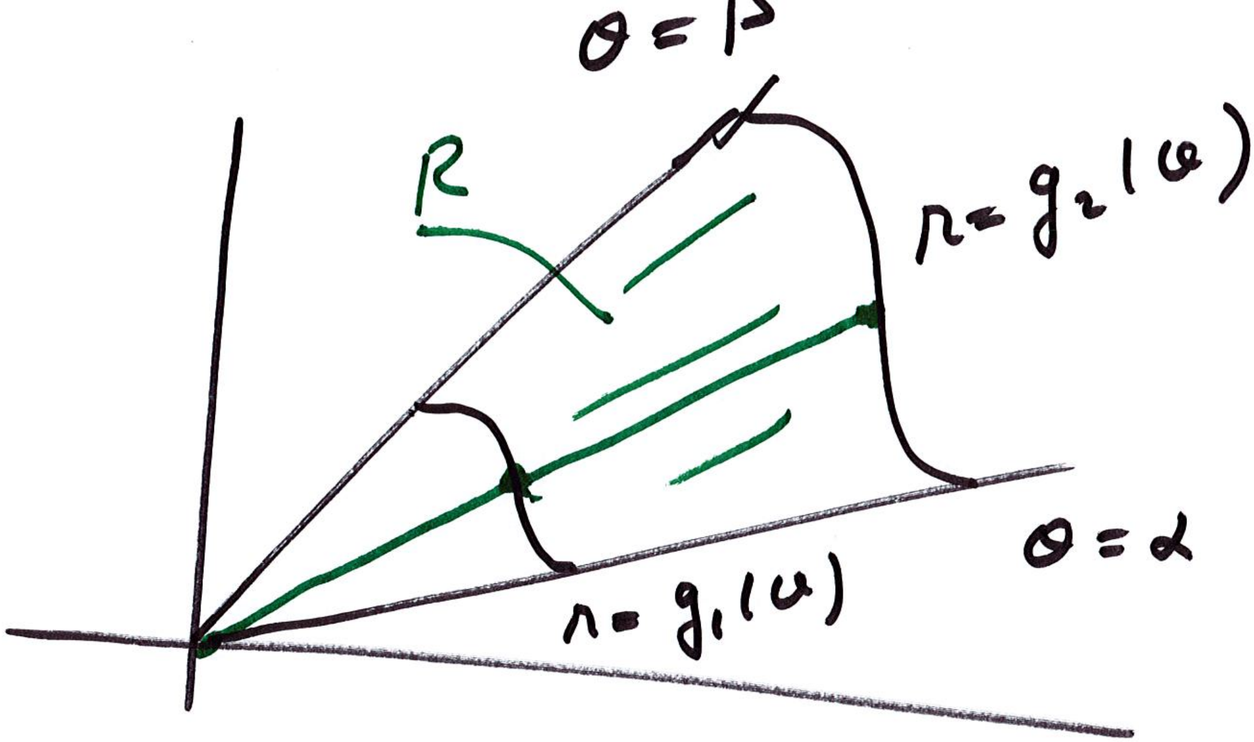
$$= \frac{1}{2}\int_0^1 e^u\,du = \frac{1}{2}\,e^u\Big|_0^1$$

$$= \frac{1}{2}(e-1).$$

$$= \int_0^{\pi/2}\frac{1}{2}(e-1)\,d\theta = \frac{1}{2}(e-1).\frac{\pi}{2}$$

$$= \frac{\pi}{4}(e-1).$$

More general regions



$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \left[\int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr \right] d\theta$$

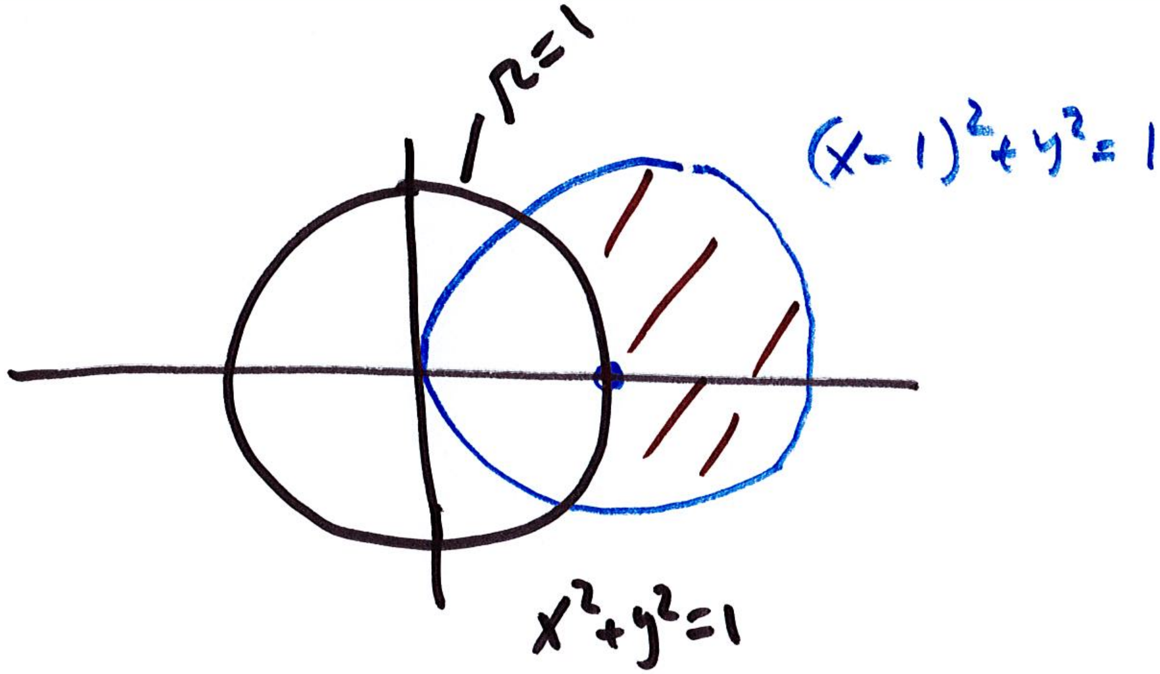
Example: Compute.

$$\iint_R x \, dA$$

R = region outside

$$x^2+y^2 \le 1$$

Inside $(x-1)^2+y^2 \le 1.$



Step 1 : Describe the region in Polar coordinates

$$x^2 + y^2 = 1 \quad , \quad r = 1$$

Region outside the circle $r \geq 1$.

$$(x-1)^2 + y^2 = 1$$

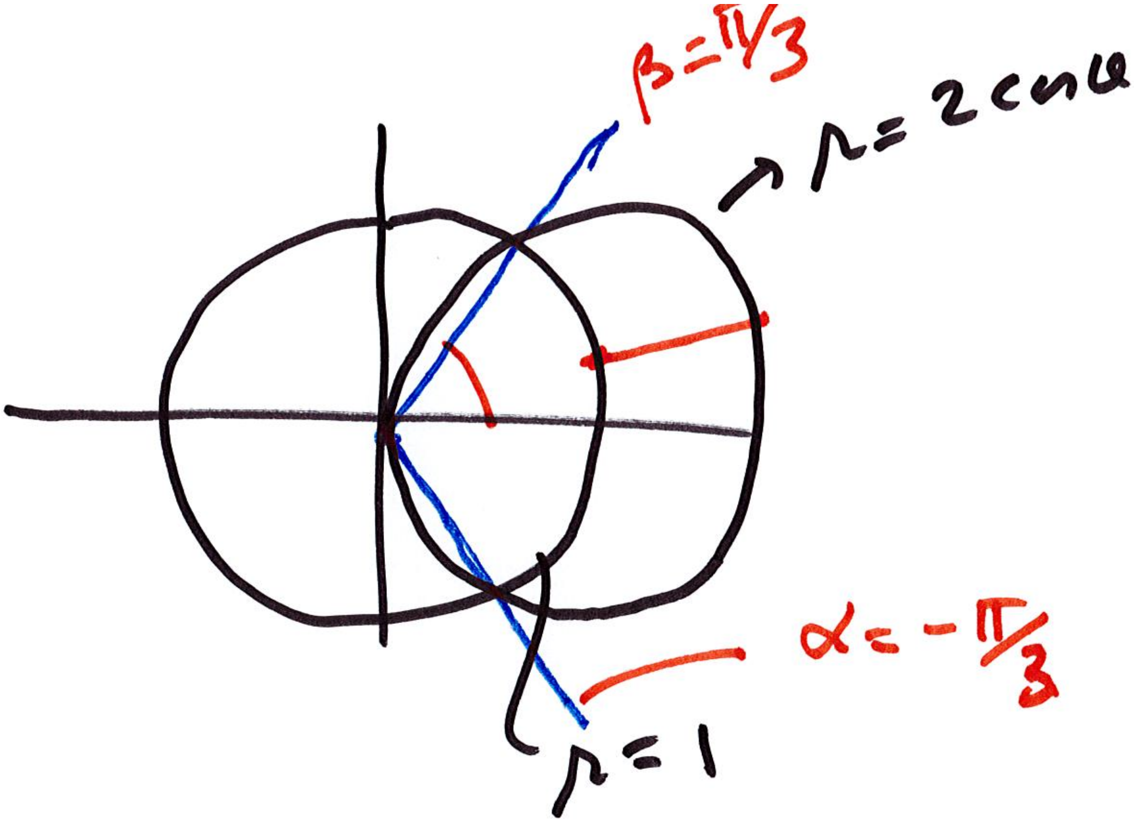
$$x^2 - 2x + 1 + y^2 = 1$$

$$\underbrace{x^2} - 2x + \underline{y^2} = 0$$

$x = r\cos\theta$, $y = r\sin\theta$. $x^2 + y^2 = r^2$

$$r^2 - 2r\cos\theta = 0$$

$$r^2 = 2r\cos\theta, \quad \boxed{r = 2\cos\theta}$$



The circles intersect

$$r = 1, \quad r = 2\cos\theta$$

$$1 = 2\cos\theta$$

$$\cos\theta = 1/2.$$

$$\iint_R x \, dA = \int_{-\pi/3}^{\pi/3} \int_1^{2\cos\theta} r\cos\theta \, r \, dr \, d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \cos\theta \left(\int_1^{2\cos\theta} r^2 \, dr \right) d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \cos\theta \left[\frac{r^3}{3} \Big|_1^{2\cos\theta} \right] d\theta.$$

$$= \frac{1}{3} \int_{-\pi/3}^{\pi/3} \cos\theta \, (8\cos^3\theta - 1) \, d\theta.$$

$$= \frac{1}{3}\int_{-\pi/3}^{\pi/3}\left[8\cos^4\alpha - \underline{\underline{\cos\alpha}}\right]d\alpha$$

$$\int_{-\pi/3}^{\pi/3}\cos^4\alpha\ d\alpha$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta - 1 + \cos^2\theta$$

$$= 2\cos^2\theta - 1$$

$$\boxed{\cos^2\theta = \frac{1+\cos 2\alpha}{2}}$$

$$= \int_{-\pi/3}^{\pi/3} \left(\cos^2 \alpha\right)^2 d\alpha$$

$$= \int_{-\pi/3}^{\pi/3} \frac{\left(1 + \cos 2\alpha\right)^2}{4} d\alpha$$

$$= \frac{1}{4} \int_{-\pi/3}^{\pi/3} \left(1 + 2\cos 2\alpha + \left(\cos 2\alpha\right)^2\right) d\alpha$$

$$= \frac{1}{4} \int_{-\pi/3}^{\pi/3} \left[1 + 2\cos 2\alpha + \left(\frac{1 + \cos 4\alpha}{2}\right)\right] d\alpha$$

Now it is easy
to compute two integrals.