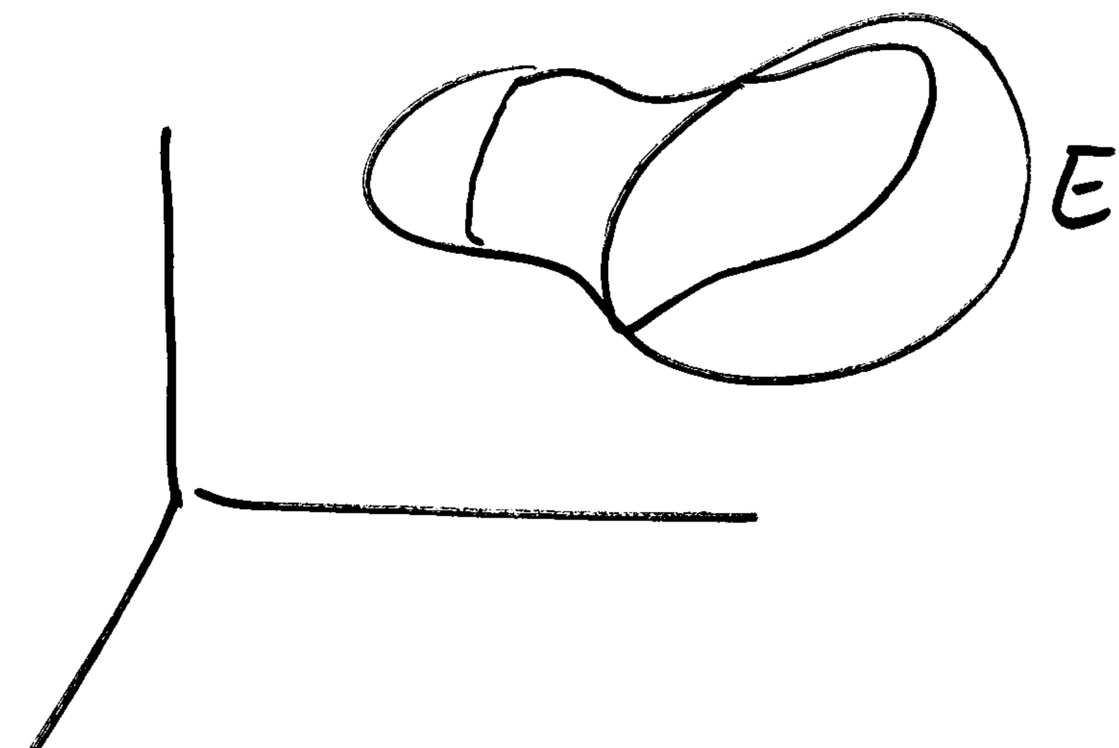
Lesson 23   Section 15.6

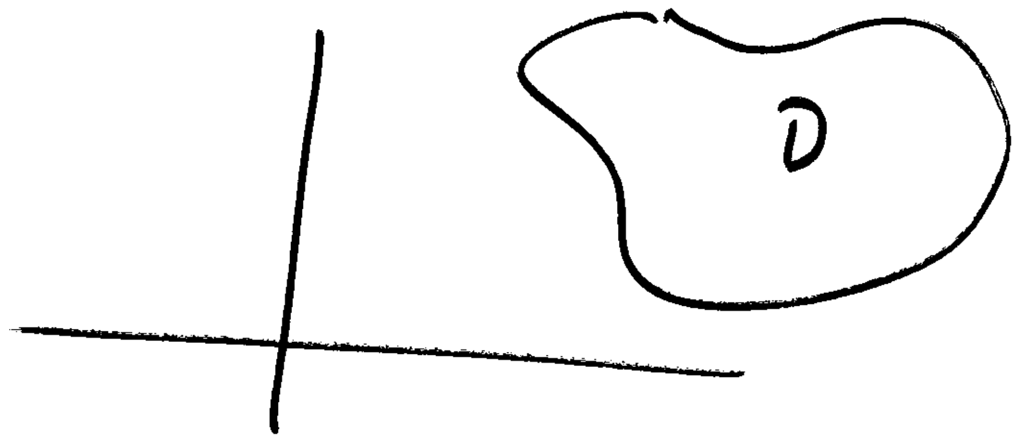
Triple Integrals

$$\iiint_E f(x,y,z)\,dV$$



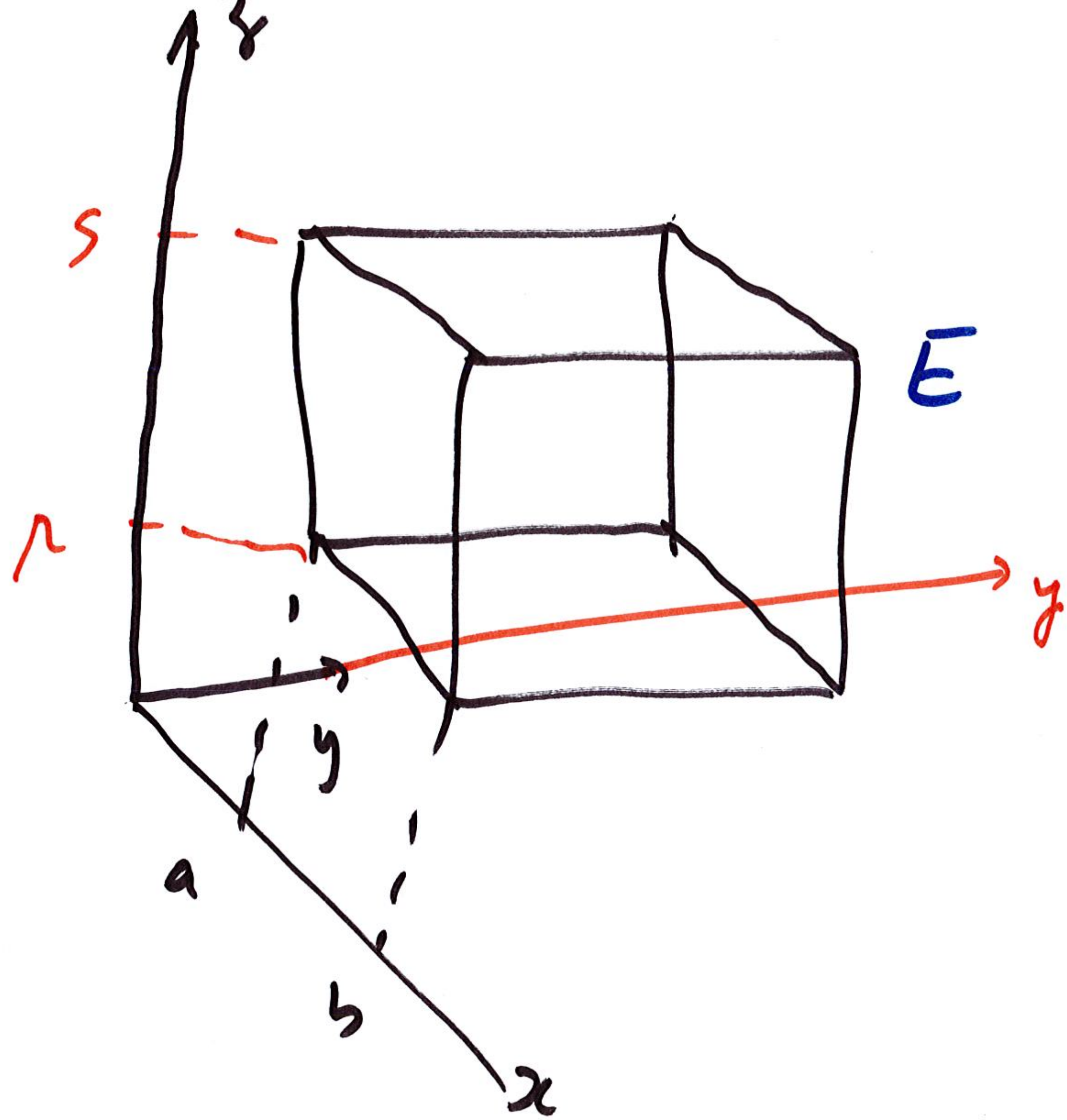
Double Integrals

$$\iint_D f(x,y)\, dA$$



I. Integrals over Boxes

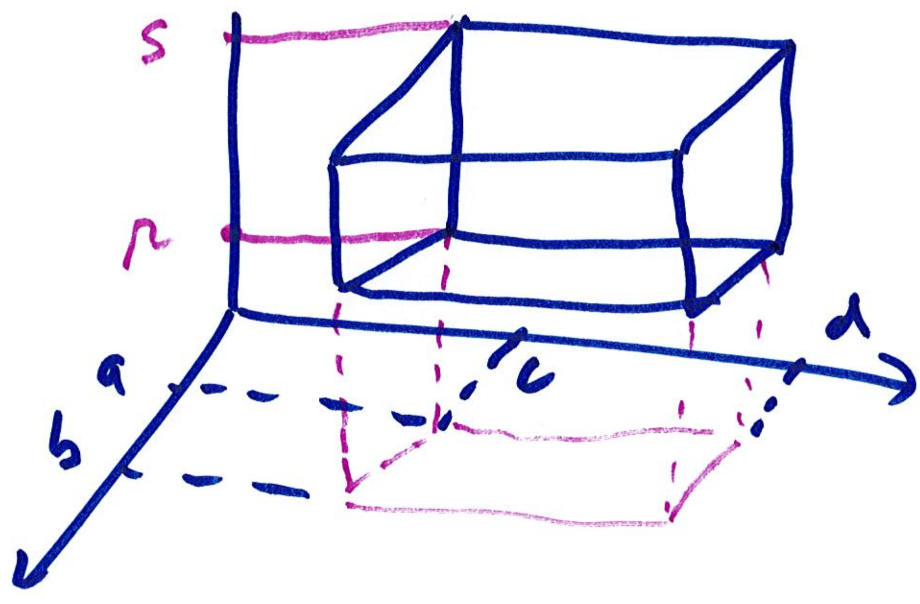
$$\text{Box} = [a,b] \times [c,d] \times [r,s]$$



$a \le x \le b$, $c \le y \le d$

$r \le z \le s.$

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_c^d \left[\int_r^s f(x,y,z)\,dz \right] dy\,dx$$



Example: Compute

$$\iiint_E xyz \, dV$$

E is the box $[0,1] \times [1,3] \times [1,2]$

$0 \leq x \leq 1$, $1 \leq y \leq 3$

$1 \leq z \leq 2$

$$\iiint_E xyz \, dV = \int_0^1 \int_1^3 \int_1^2 xyz \, dz \, dy \, dx$$

$$= \int_0^1 \int_1^3 xy \left(\frac{z^2}{2} \Big|_1^2 \right) dy \, dx$$

$$= \int_0^1 \int_1^3 \frac{3}{2} xy \, dy \, dx$$

$$= \int_0^1 \frac{3}{2} x \cdot \left(\frac{y^2}{2} \Big|_1^3 \right) dx$$

$$= \int_0^1 \frac{3}{2} x \cdot \left(\frac{8}{2} \right) dx$$

$$= \int_0^1 6x \, dx = 3x^2 \Big|_0^1 = 3.$$

$$E = [a,b] \times [c,d] \times [r,s]$$

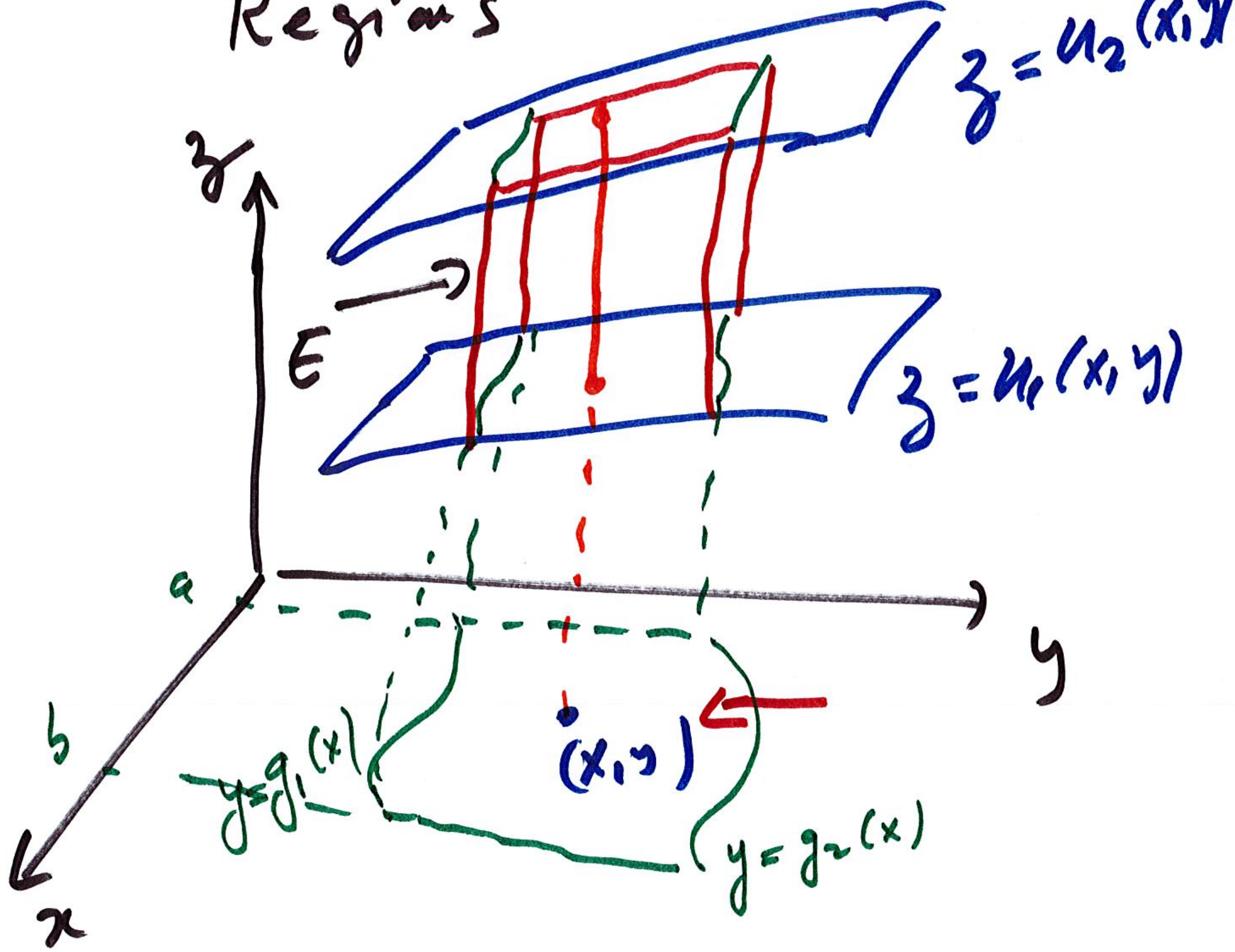
$$\iiint_E f(x,y,z)\, dV =$$

$$= \int_a^b \int_c^d \int_r^s f(x,y,z)\, dz\, dy\, dx$$

$$= \int_r^s \int_a^b \int_c^d f(x,y,z)\, dy\, dx\, dz$$

$$= \int_r^s \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz$$

Integration over general Regions

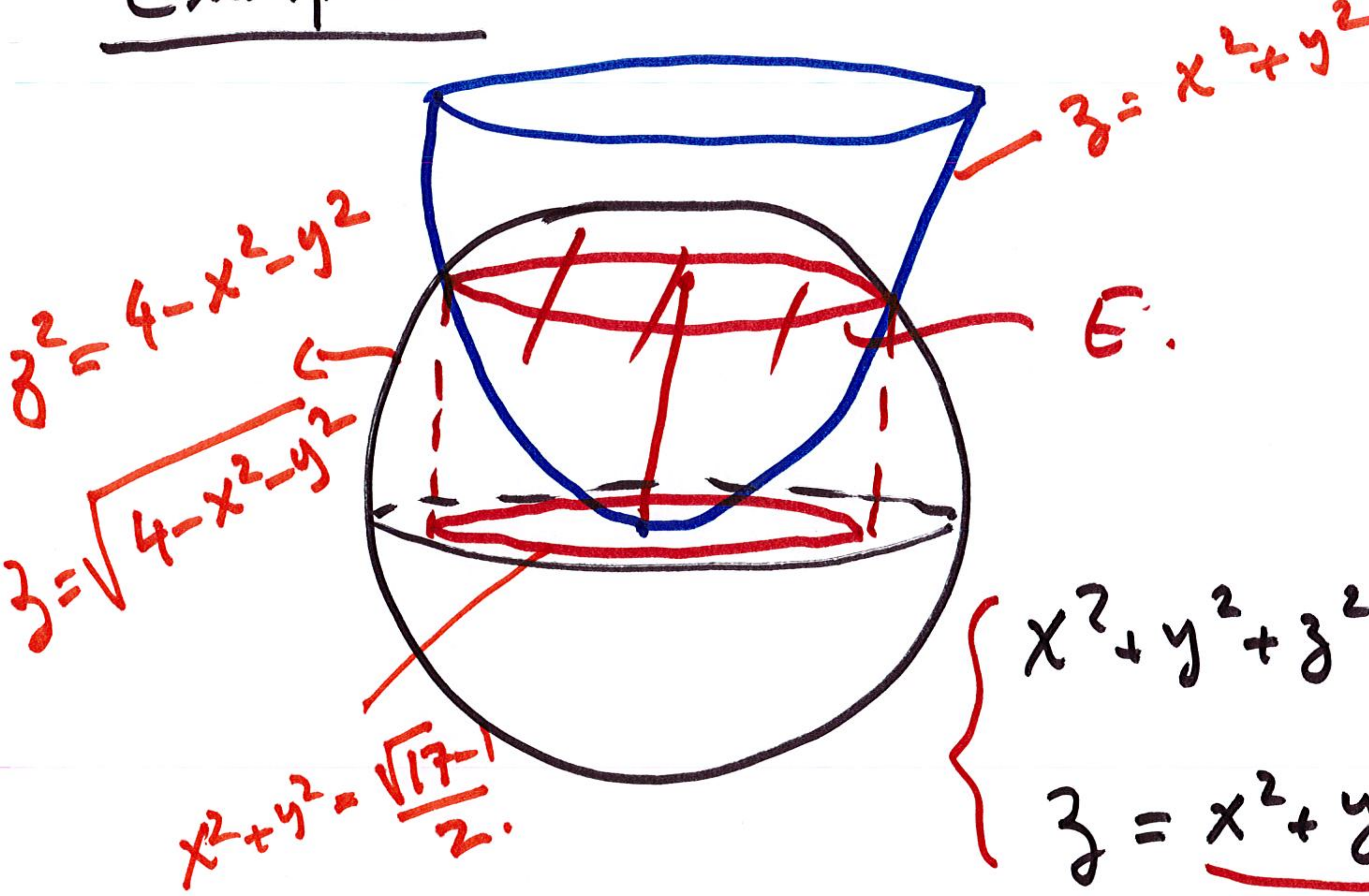


Fixed (x,y) $\quad u_1(x,y) \le z \le u_2(x,y)$

$$g_1(x) \le y \le g_2(x)$$

$$a \le x \le b.$$

Example:



$$\begin{cases} x^2 + y^2 + z^2 = 4 \\ z = \underline{x^2 + y^2} \end{cases}$$

$$\underbrace{x^2 + y^2}_{u_1(x,y)} \le z \le \underbrace{\sqrt{4 - x^2 - y^2}}_{u_2(x,y)}.$$

$$z = x^2 + y^2 \; ; \quad x^2 + y^2 = 4 - z^2$$

$$z = 4 - z^2$$

$$z^2 + z - 4 = 0$$

$$z = \frac{-1 \pm \sqrt{1 - 4(-4)}}{2}$$

$$z = \frac{-1 + \sqrt{17}}{2} \quad , \cancel{z = \frac{-1 - \sqrt{17}}{2}}$$

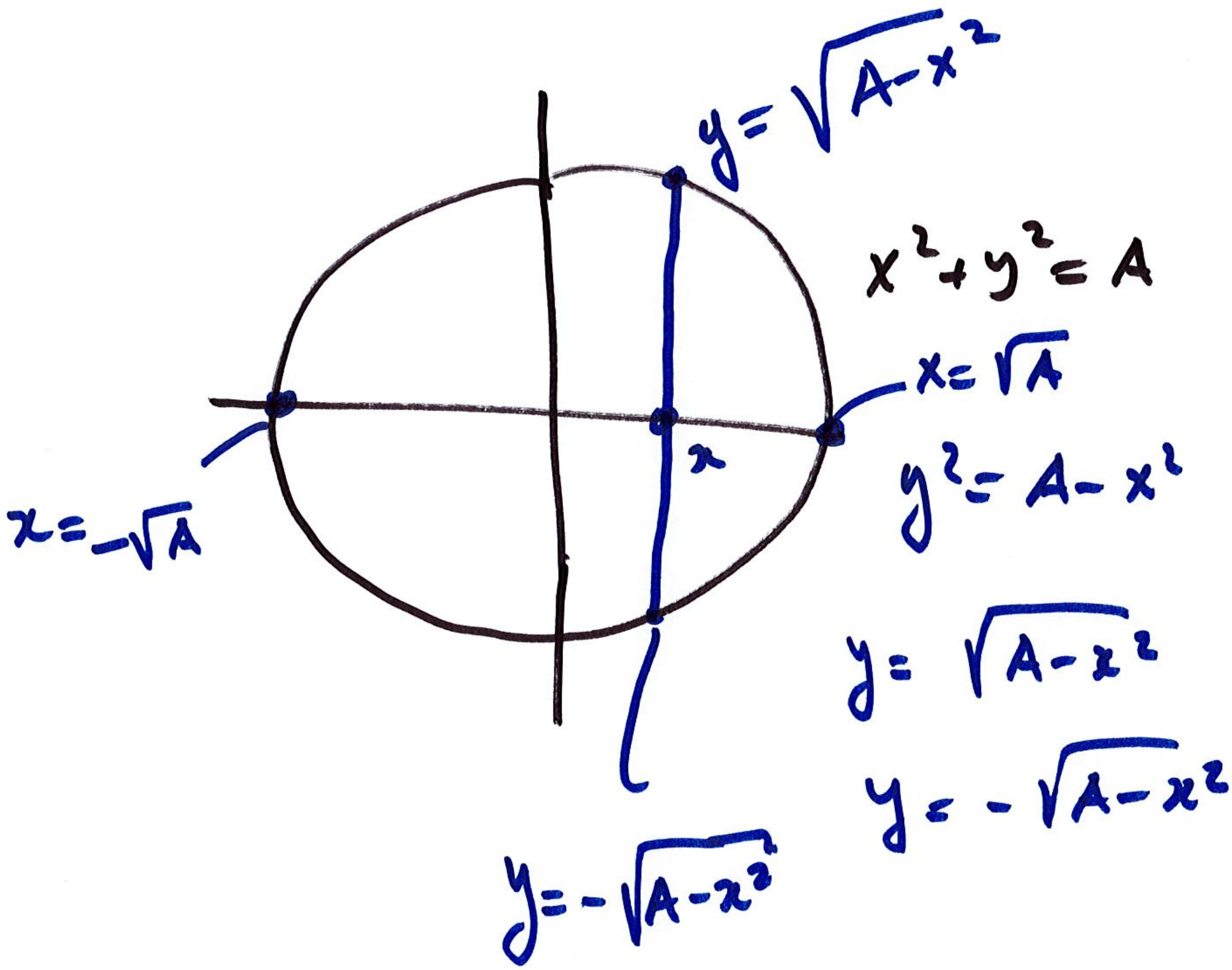
We want $z > 0$.

$$\boxed{z = \frac{-1 + \sqrt{17}}{2}}$$

$$x^2 + y^2 = z$$

$$\boxed{x^2 + y^2 = \frac{\sqrt{17} - 1}{2}}$$

$$\boxed{x^2 + y^2 = \frac{\sqrt{17} - 1}{2} = A}$$



$$-\sqrt{A - x^2} \leq y \leq \sqrt{A - x^2}.$$

$$-\sqrt{A} \leq x \leq A.$$

If the region E is described by

$$u_1(x,y) \le z \le u_2(x,y)$$

$$g_1(x) \le y \le g_2(x)$$

$$a \le x \le b$$

$$\iiint_E f(x,y,z)\,dV =$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dy\,dx$$

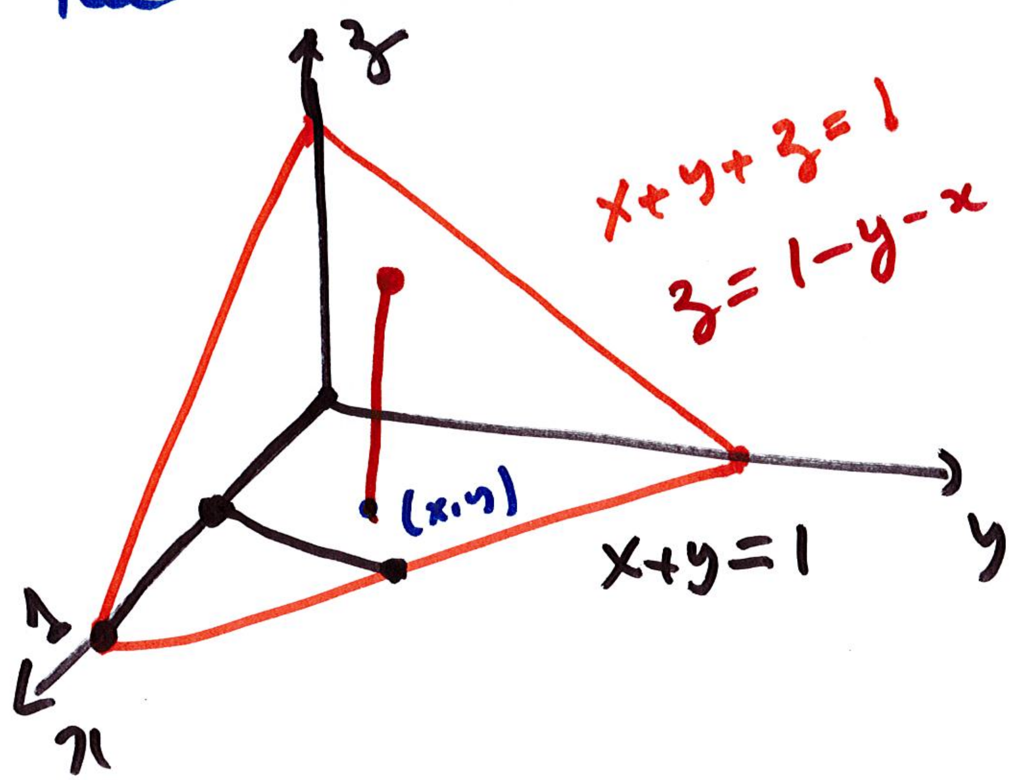
Evaluate

$$\iiint_E z \, dV$$

E is the region of the 1st octant bounded by

$$x + y + z = 1$$

and the coordinate planes.



We need to describe

the region E

$$u_1(x,y) \le z \le u_2(x,y)$$

$$0 \le z \le 1-y-x$$

$$0 \le y \le 1-x$$

$$0 \le x \le 1.$$

So

$$\iiint_E z \, dV = \int_0^1 \int_0^{1-x} \left[\int_0^{1-y-x} z \, dz \right] dy$$

$$= \int_0^1 \underbrace{\int_0^{1-x} \frac{1}{2}(1-y-x)^2\, dy}\, dx$$

$$\int_0^{1-x} (1-x-y)^2\, dy$$

Set

$u = 1-x-y$

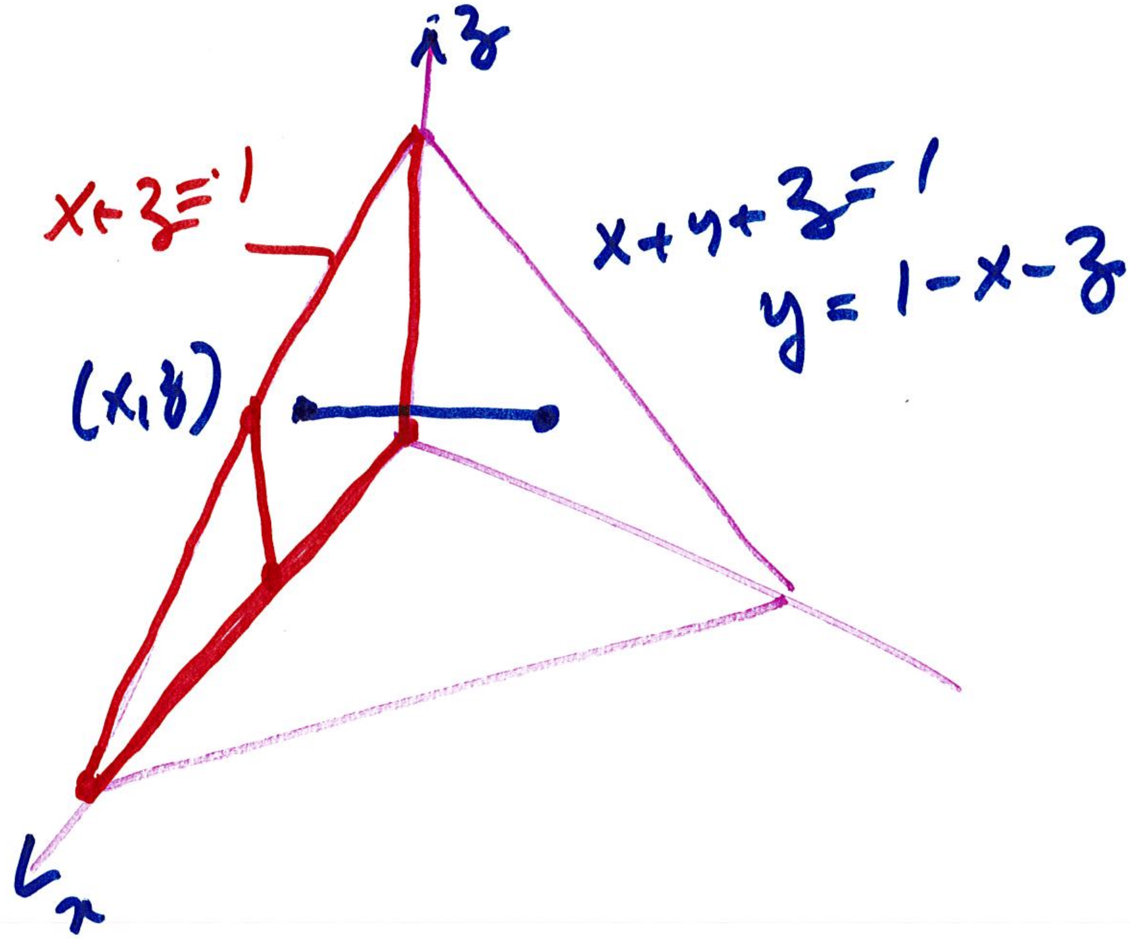
$du = -dy$

$$= \int_{1-x}^{0} u^2 \quad (-du)$$

$$= \int_0^{1-x} u^2\, du = \frac{1}{3}(1-x)^3$$

$$= \int_0^1 \frac{1}{6}(1-x)^3\, dx . \quad 1-x=v$$

$$= 1/24 .$$



$$\int \int \int z \, dy \, dz \, dx$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-z} z \, dy \, dz \, dx$$