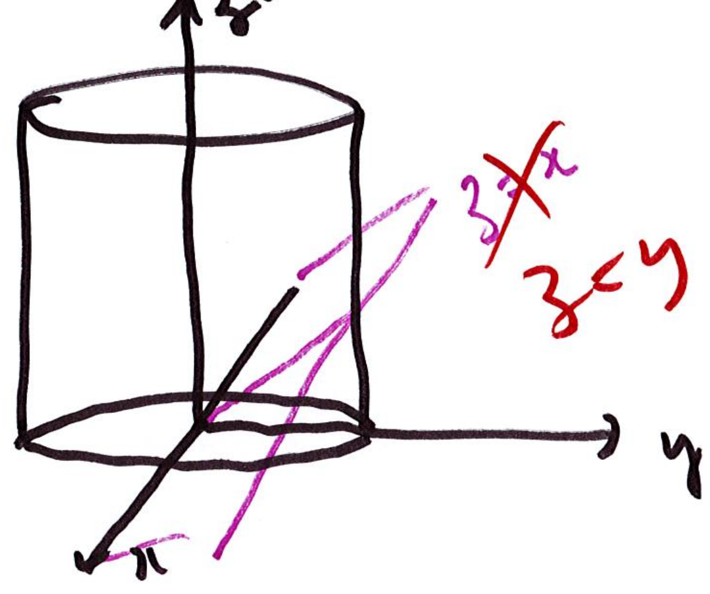
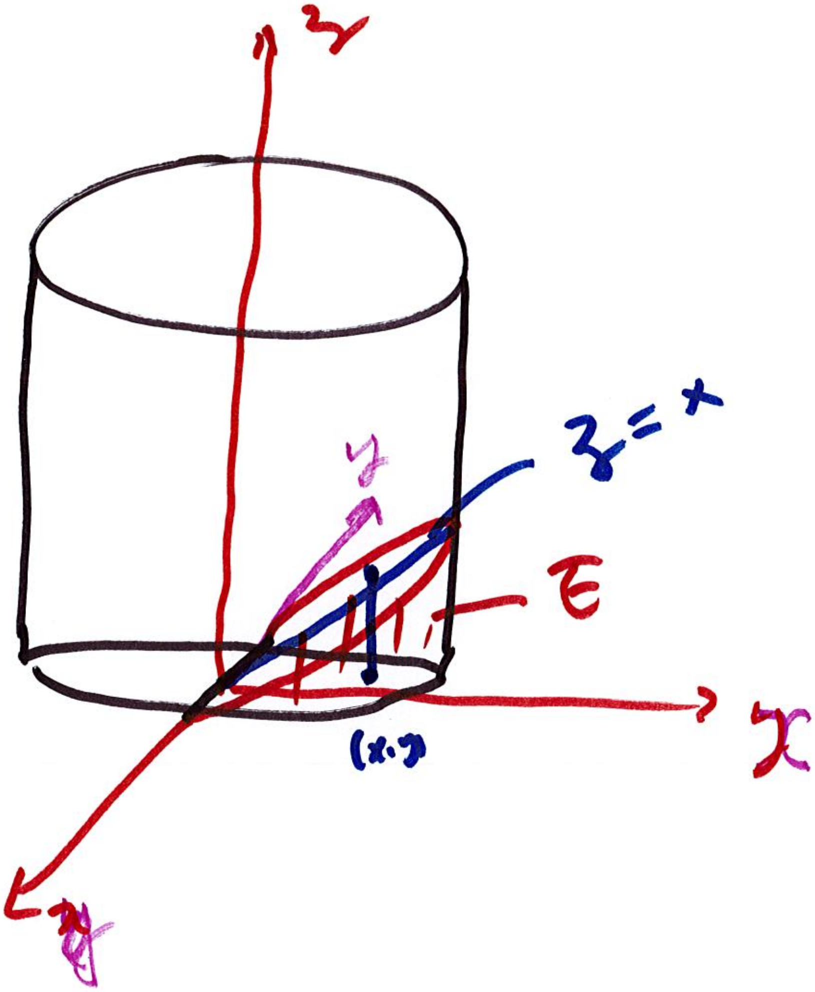
Lesson 24 Section 15.7

Triple Integrals in Cylindrical Coordinates.

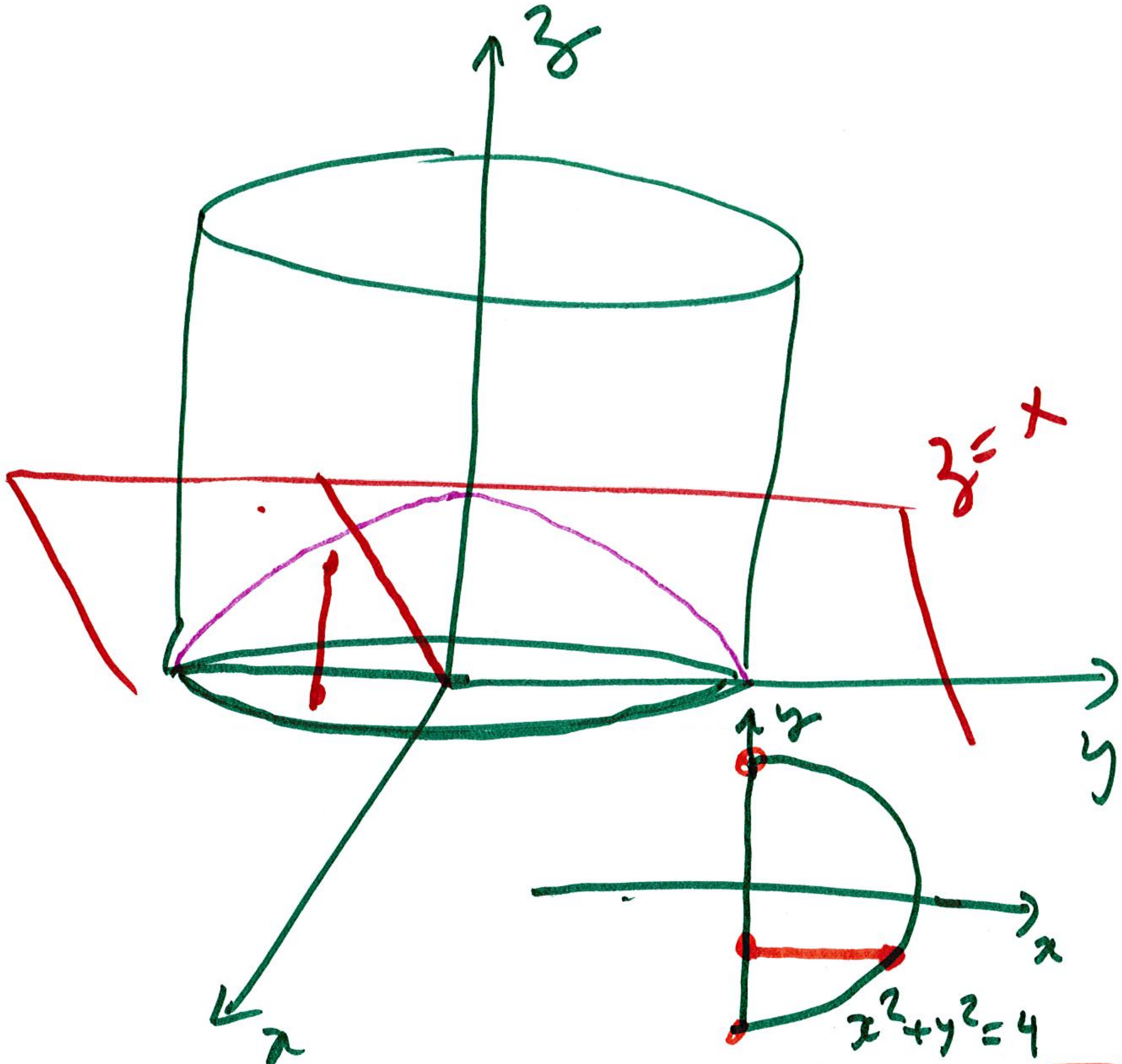
Example: Compute $\iiint_E z \, dV$

E is the region bounded by $x^2+y^2=4$, $z=x$ and the xy-plane.





$$\iiint_E z\, dV = \int \int \int_0^x z\, dz$$



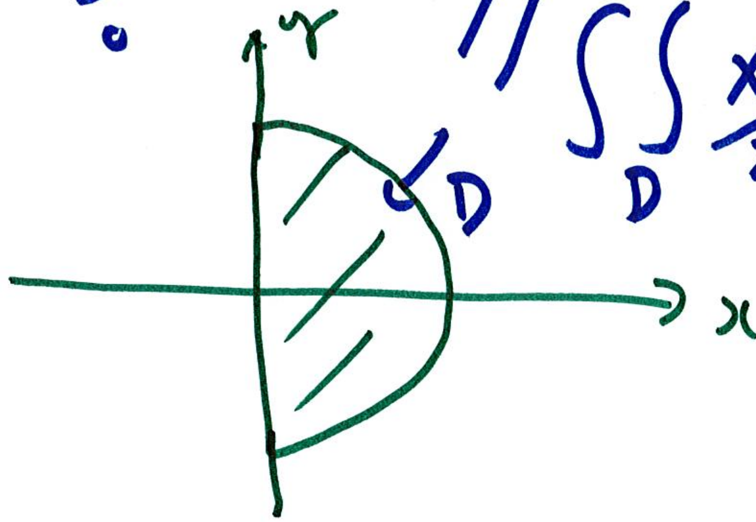
$x = \sqrt{4-y^2}$

$$\iiint_E z\,dV = \int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{0}^{x} z\,dz\,dx\,dy$$

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\underbrace{\int_{0}^{x} z\,dz}\,dx\,dy$$

$$= \int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} \frac{x^2}{2}\,dx\,dy$$

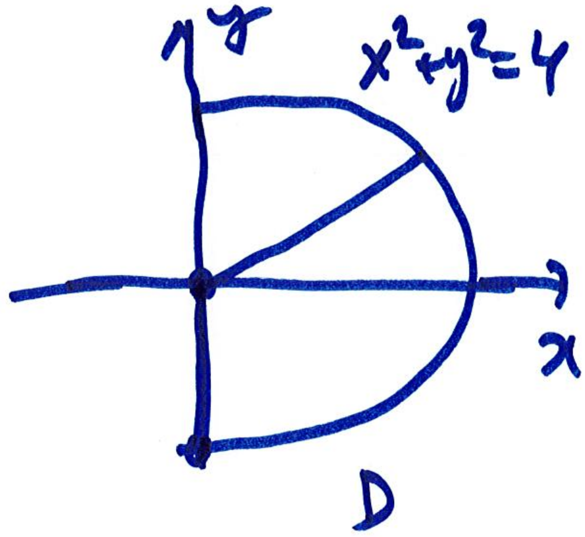
$$= \iint_D \frac{x^2}{2}\,dA$$



I will use polar coordinates to compute this integral.

$$x = r\cos\theta$$

$$x^2 = r^2\cos^2\theta$$



$$\iint_D \frac{x^2}{2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 \frac{r^2\cos^2\theta}{2}\, r\,dr\,d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2}\int_0^2 r^3\cos^2\theta\,dr\,d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} 4\cos^2\theta\,d\theta \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}.$$

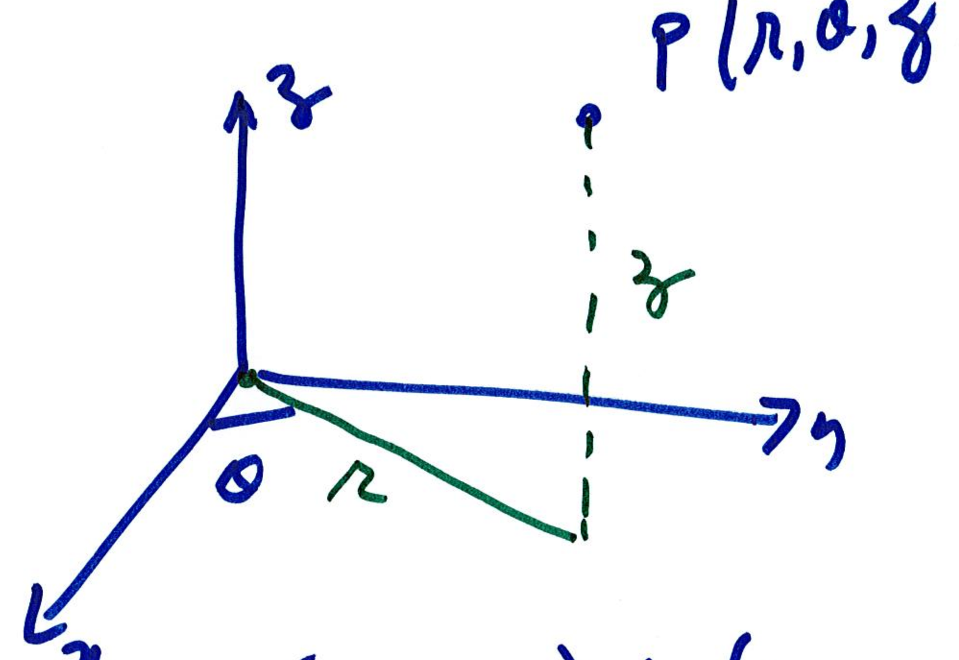
Leave it for you.

The method was:

(1) Integrated in z

(2) Used polar coordinates to integrate in x and y.

Cylindrical coordinates



(r, θ) polar coordinates on the xy plane.

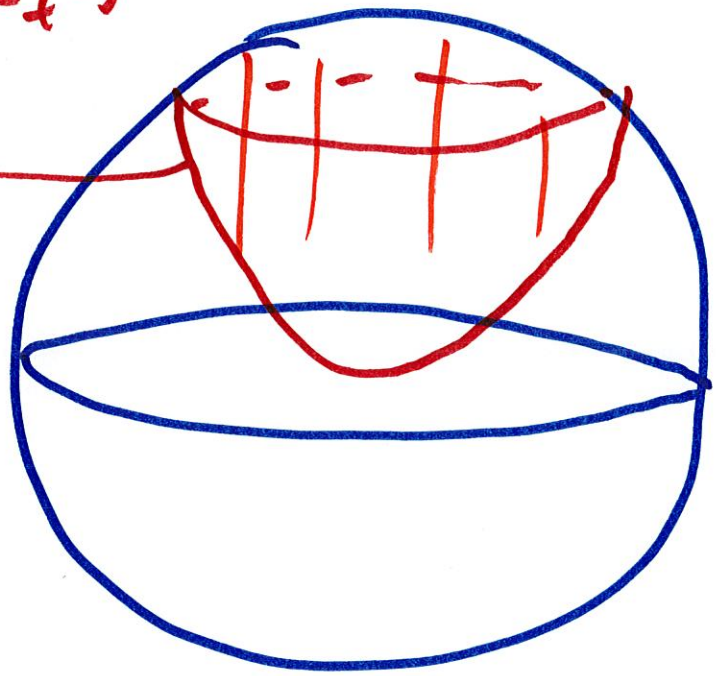
$$\iiint_E f \, \underbrace{dV}_{} =$$

$$= \iiint f \, \underset{=}{dz} \, \underbrace{r \, dr \, d\theta}_{}$$

<u>Examples:</u>

1) Compute the volume of the region above the paraboloid $z = x^2 + y^2$ inside the sphere $x^2 + y^2 + z^2 = 2$.

$z = x^2 + y^2$



$\underbrace{x^2 + y^2}_{r^2} + z^2 = 2$

$r^2 + z^2 = 2.$

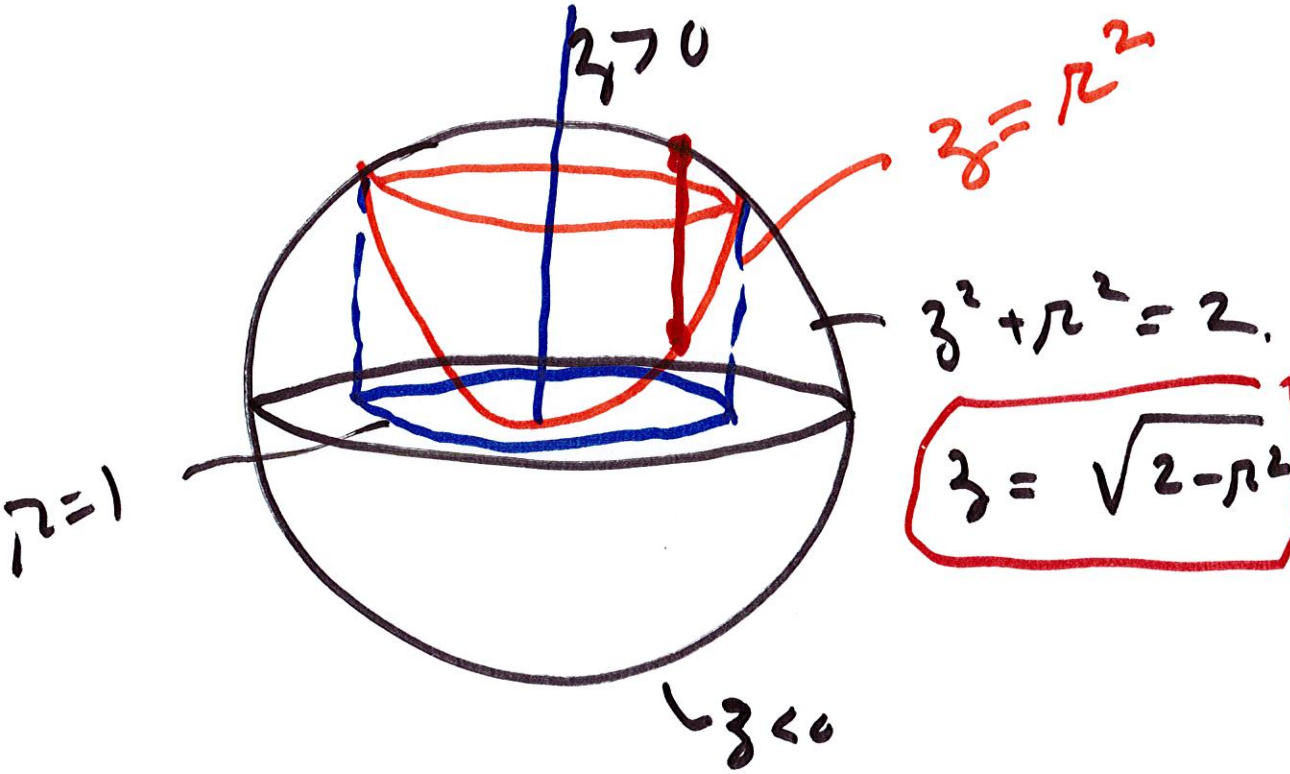
Use cylindrical coordinates

(z, r, θ).

1) Express the surfaces in cylindrical coordinates.

$x = r\cos\theta, \quad y = r\sin\theta$

$x^2 + y^2 = r^2.$



$$\boxed{z = \sqrt{2-r^2}}$$

$$\iiint_E dV = \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} dz\, \underline{\underline{r}}\, dr\, d\theta$$

The two surfaces intersect

when $z = r^2$ $\quad \underset{\underset{z}{\parallel}}{r^2} + z^2 = 2$

$z + z^2 = 2 \qquad z^2 + z - 2 = 0$

$(z+2)(z-1) = 0$

roots ~~r~~ $z=1$, $z=-2$

But $z>0$

So $z=1$ is

the one we want.

$r^2 = z$, $r^2=1$ $\boxed{r=1}$

So the volume is given by

$$\int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(\sqrt{2-r^2} - r^2\right) r\, dr\, d\theta.$$

$= \dots$

Exercise.

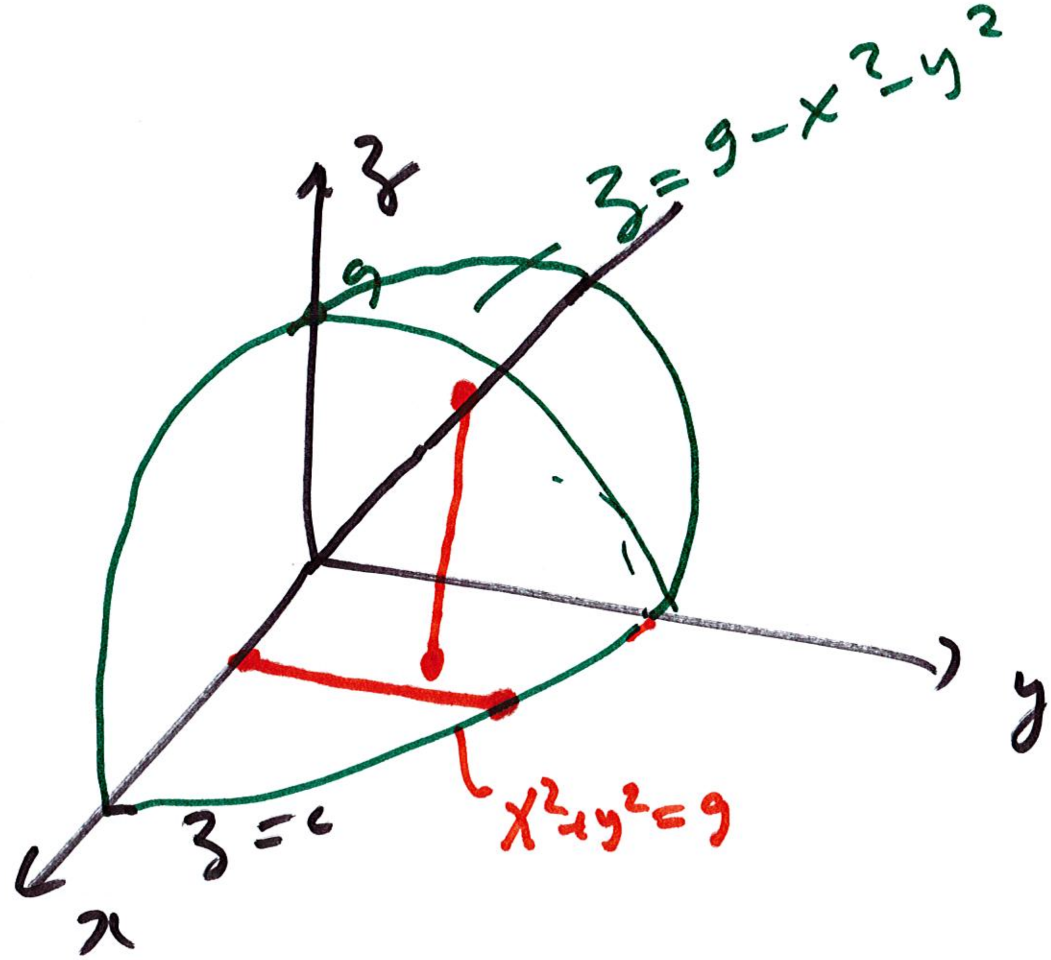
Evaluate the following integral by changing to cylindrical coordinates.

$$\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{0}^{9-x^2-y^2} \sqrt{x^2+y^2}\, dz\, dy\, dx$$

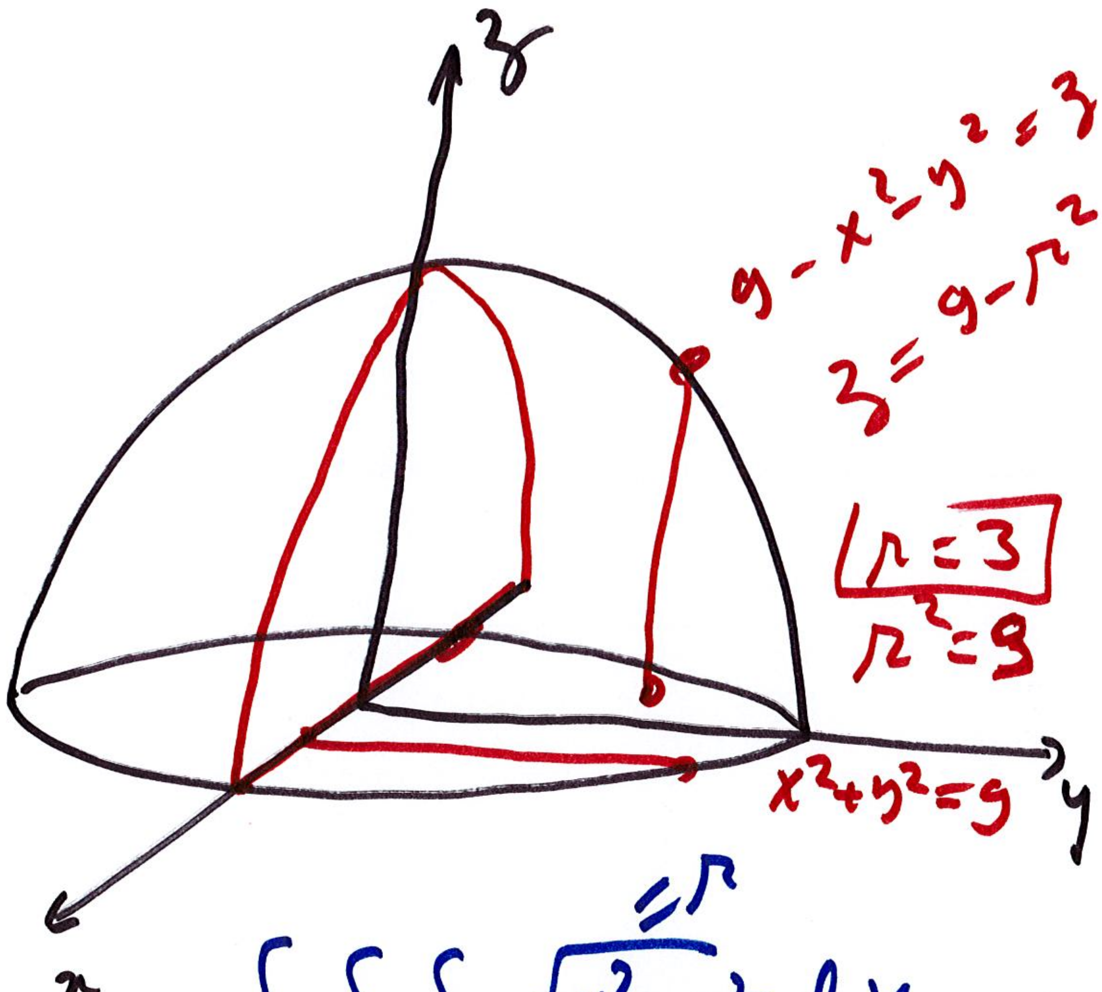
$$= \iiint_E \sqrt{x^2+y^2}\, dV.$$

$$= \int_{?}^{?} \int_{?}^{?} \int_{?}^{?} r\, dz\, r\, dr\, d\theta.$$

$$0 \le z \le 9 - x^2 - y^2$$



$$z = 9 - x^2 - y^2$$



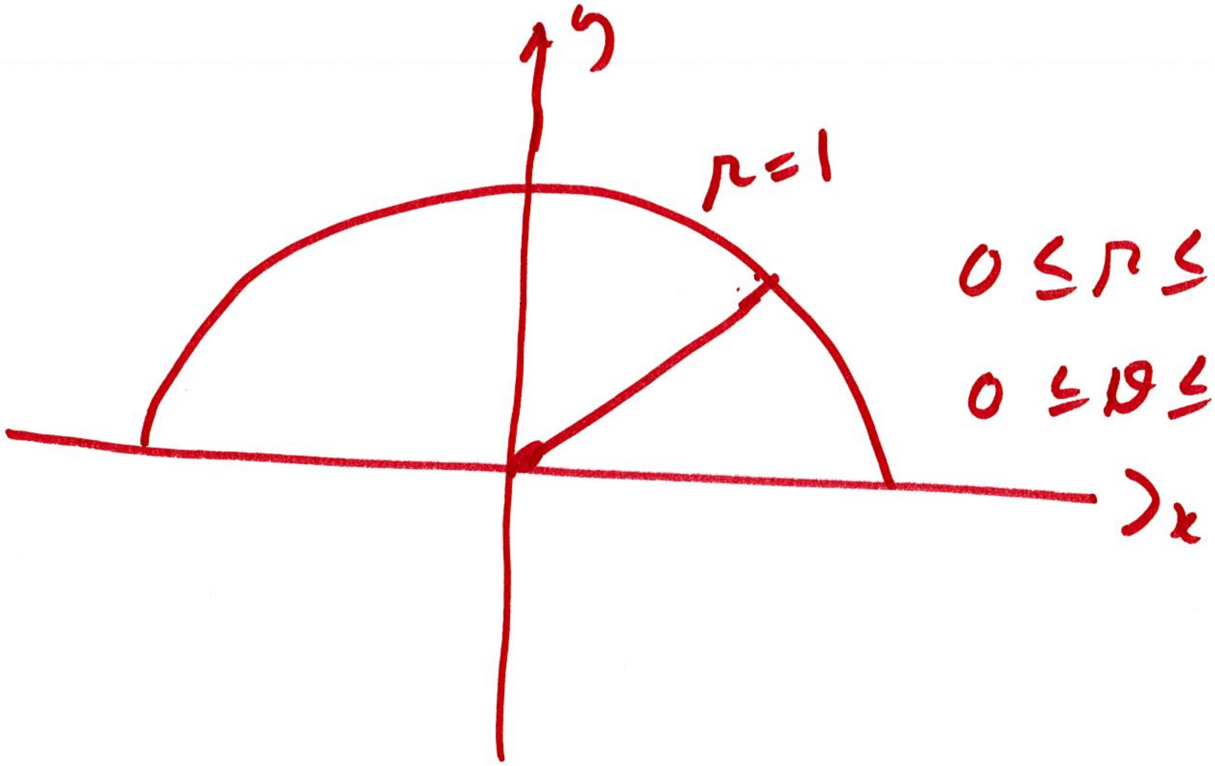
$$\iiint_E \underbrace{\sqrt{x^2+y^2}}^{=r}\, dV$$

$$\int_0^{\pi}\int_0^{3}\int_0^{9-r^2} r\, dz\; r\, dr\, d\theta$$

$$= \int_0^{\pi}\int_0^{3} r^2(9-r^2)\, dr\, d\theta$$

$= \ldots$

Exercise.



$0 \le r \le 1$

$0 \le \theta \le \pi$