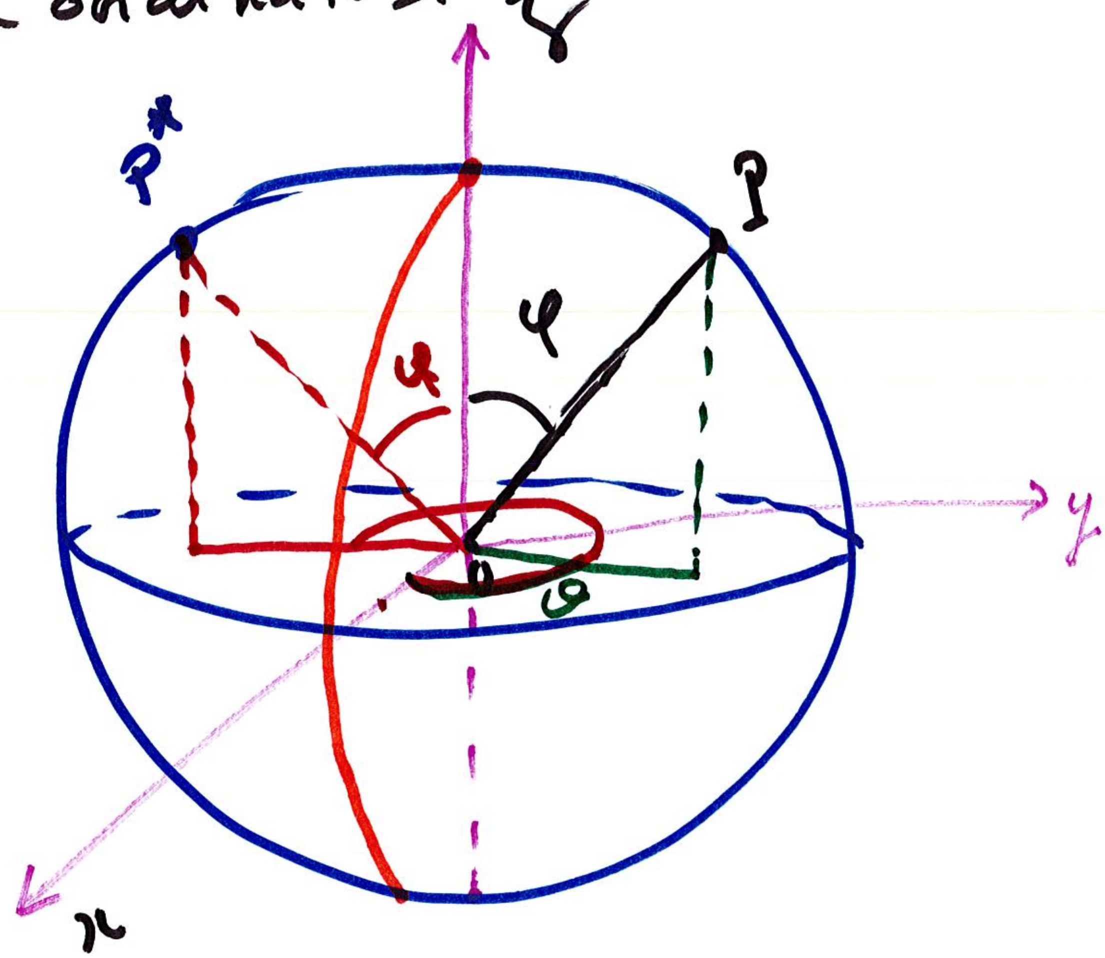
# Lesson 25 Section 15.8

Triple Integrals in Spherical Coordinates.

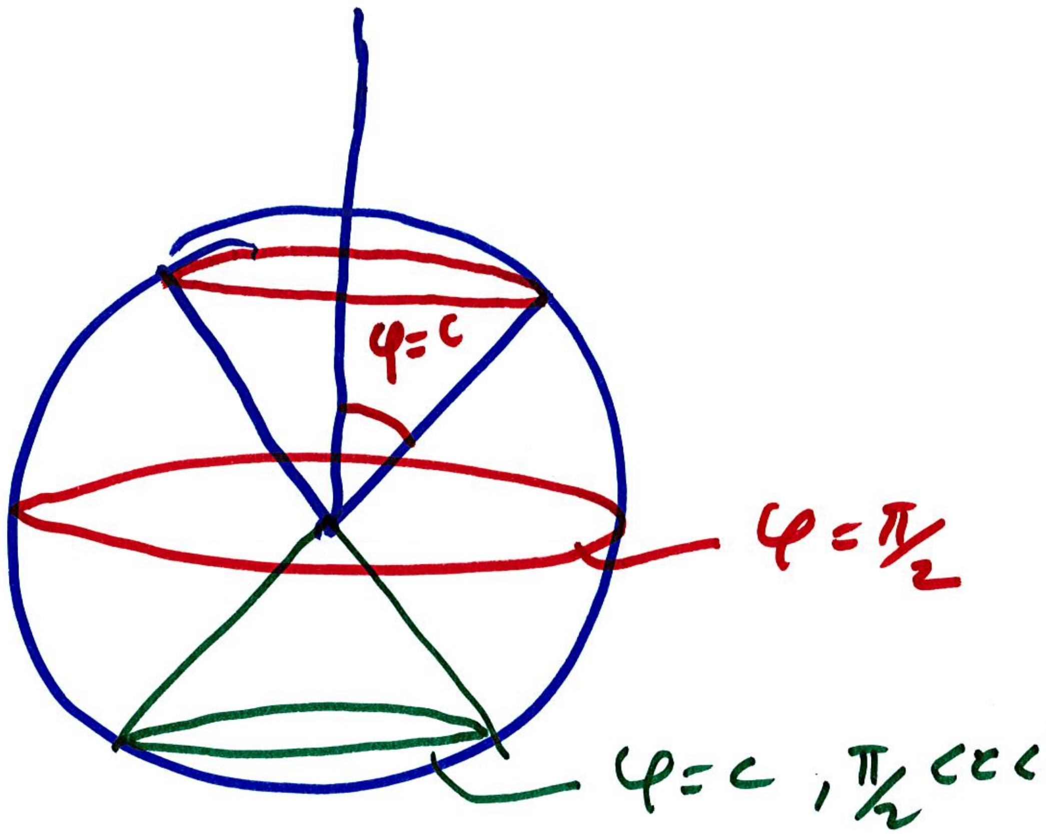



$\varphi$ = Angle the segment OP makes with the $z$-axis

$\theta$ = Angle that the projection of $\overline{OP}$ to $xy$-plane makes with the $x$-axis.

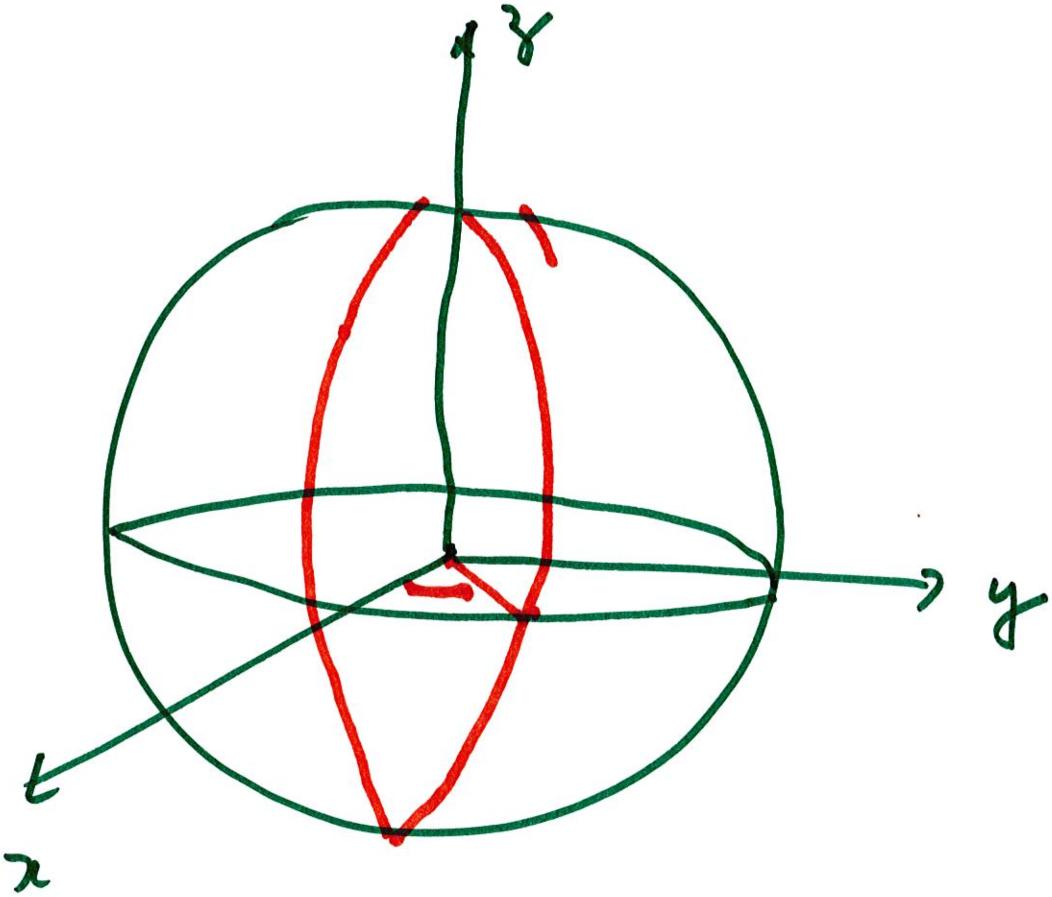
$0 \leq \varphi \leq \pi$

$0 \leq \theta \leq 2\pi$

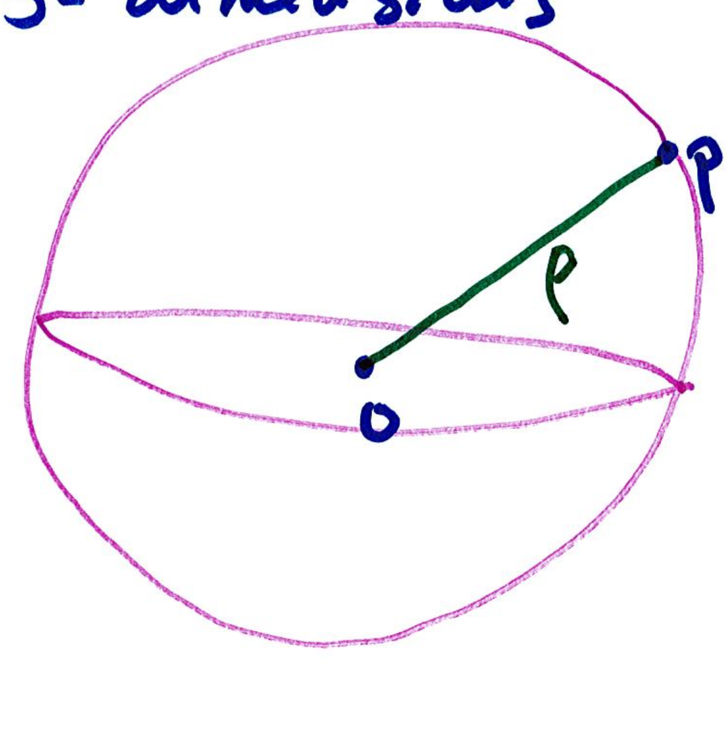



$\varphi = c \qquad 0 < c < \pi/2$

$\varphi = \pi/2$



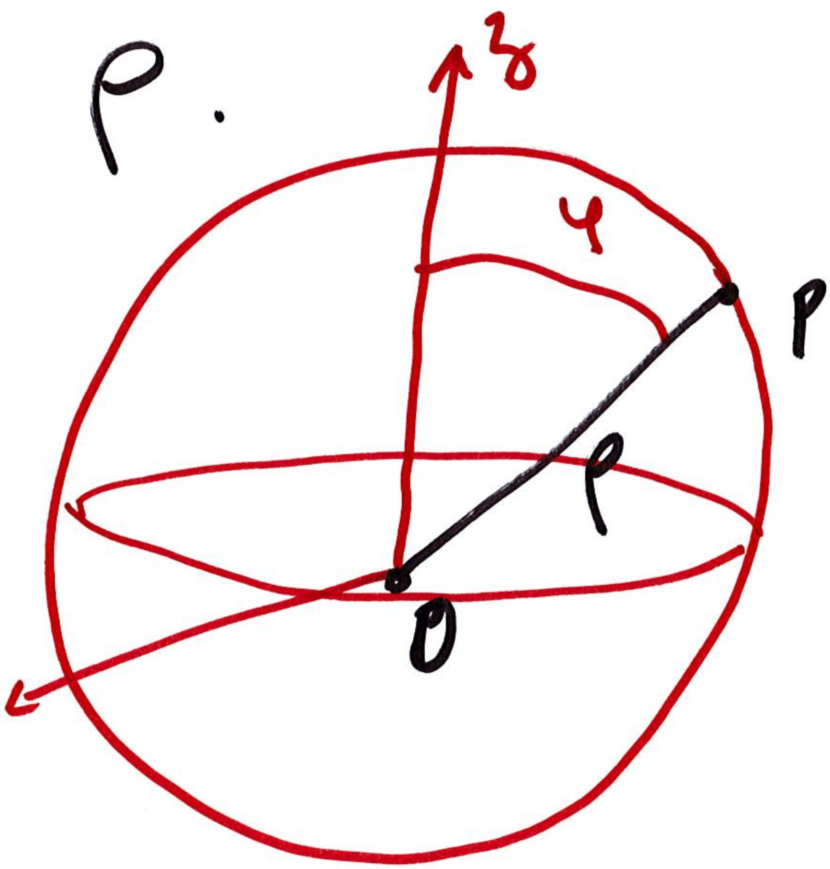

Spherical Coordinates in 3-dimensions

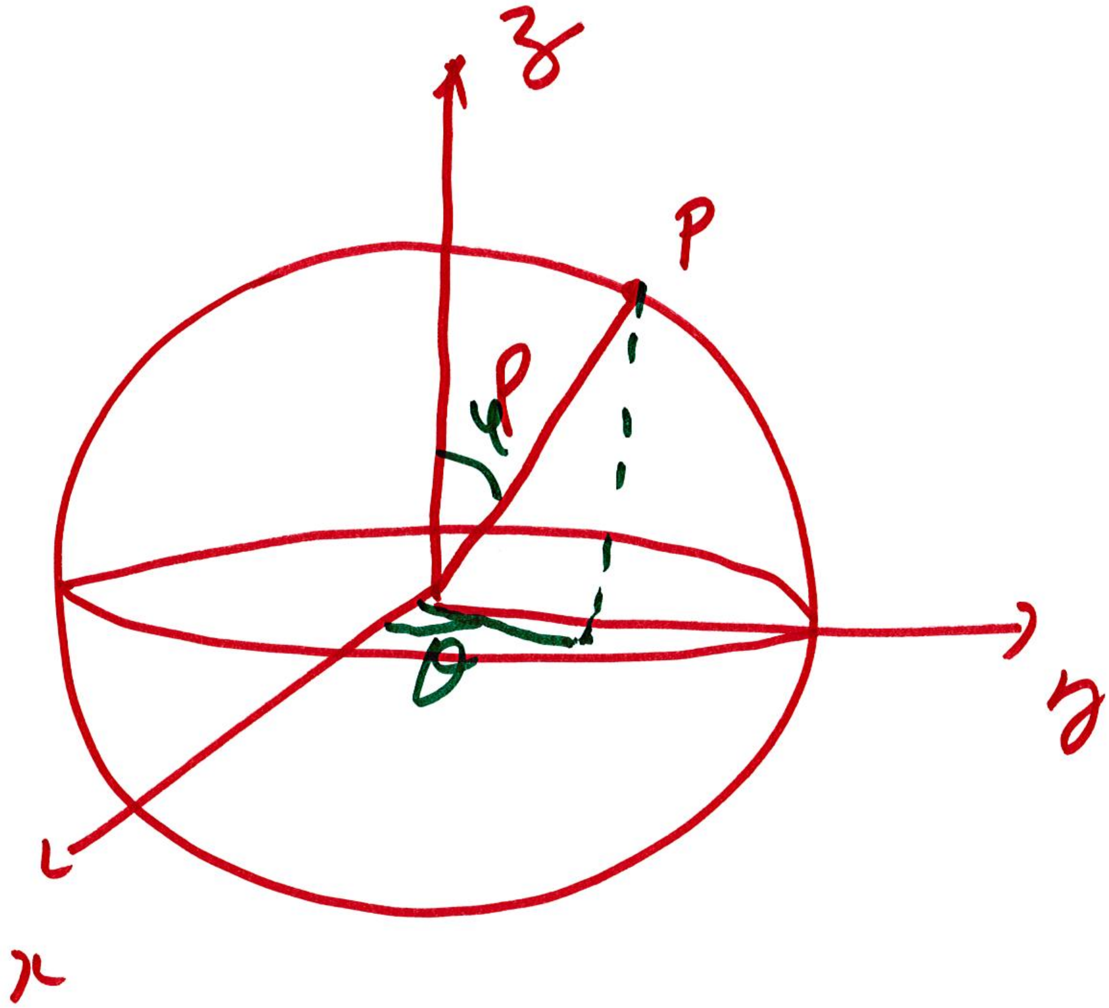


$P(\underline{\underline{\rho}}, \underline{\underline{\theta}}, \underline{\underline{\varphi}})$

$\rho$ = distance from $P$ to $O$

$\theta, \varphi$ are spherical coordinates on the sphere of radius $\rho$.






From spherical to rectangular

Coordinates.

Express $P(1, \pi/3, \pi/6)$ in rectangular coordinates.

$$P(\rho, \theta, \varphi)$$

$$\rho = 1, \quad \theta = \pi/3, \quad \varphi = \pi/6.$$

$$z = \rho \cos\varphi \; ; \quad z = 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}.$$

$$x = \rho \underbrace{\sin\varphi} \cdot \cos\theta = 1 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$y = \rho \sin\varphi \cdot \sin\theta = 1 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}.$$

$$P\left(1/4, \sqrt{3}/4, \sqrt{3}/2\right).$$

# Integration In spherical Coordinates

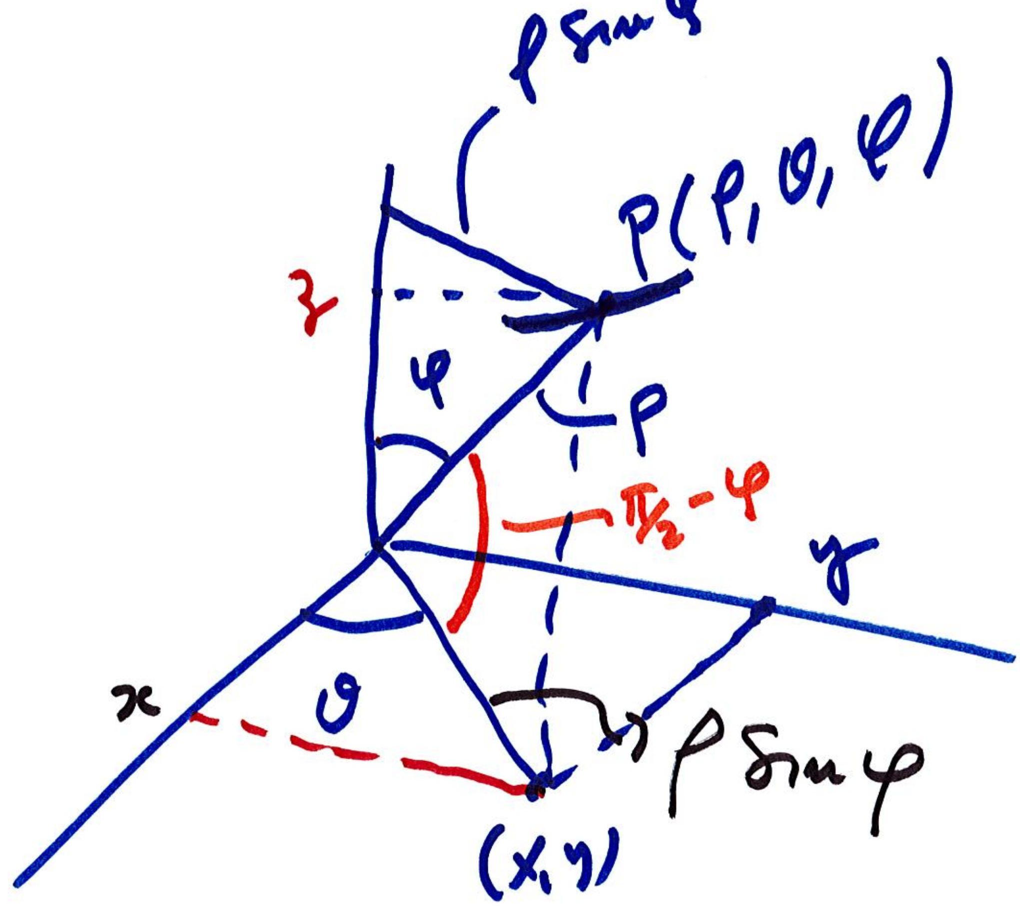
Spherical "parallelepiped."

Spherical wedge

$$a \le \rho \le b$$

$$\alpha \le \theta \le \beta$$

$$c \le \varphi \le d$$

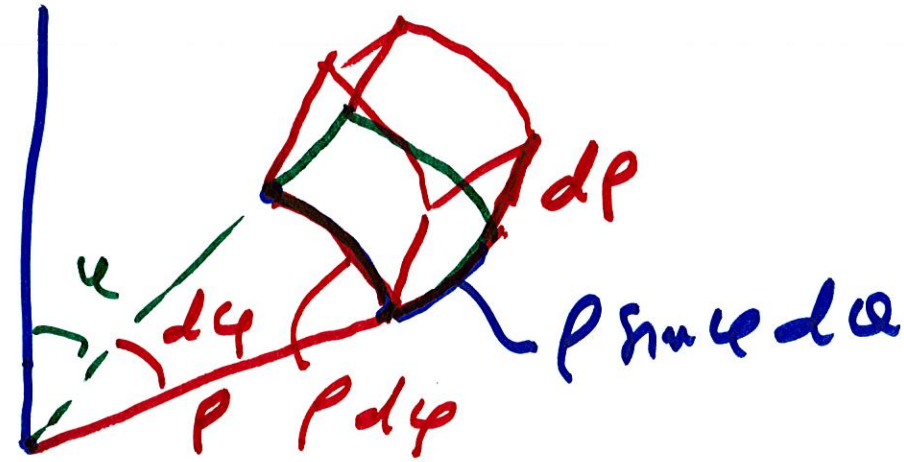



$$z = \rho \cos \varphi \qquad \rho \sin\varphi = \sqrt{x^2 + y^2}.$$

$$x = \rho \sin\varphi \cos\theta$$

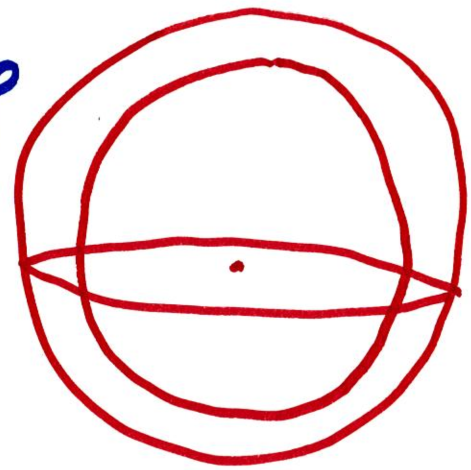
$$y = \rho \sin\varphi \sin\theta.$$

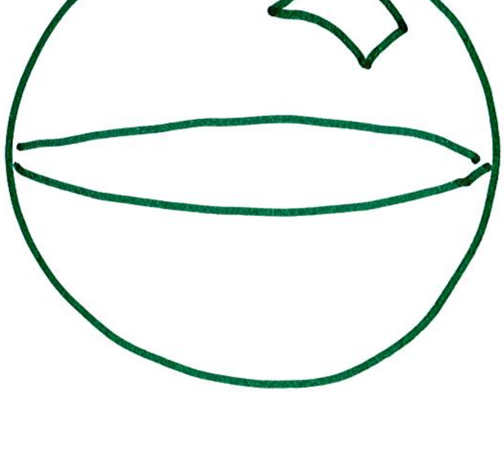
$$\rho = \sqrt{x^2 + y^2 + z^2}$$




Volume of such wedge

$= \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$



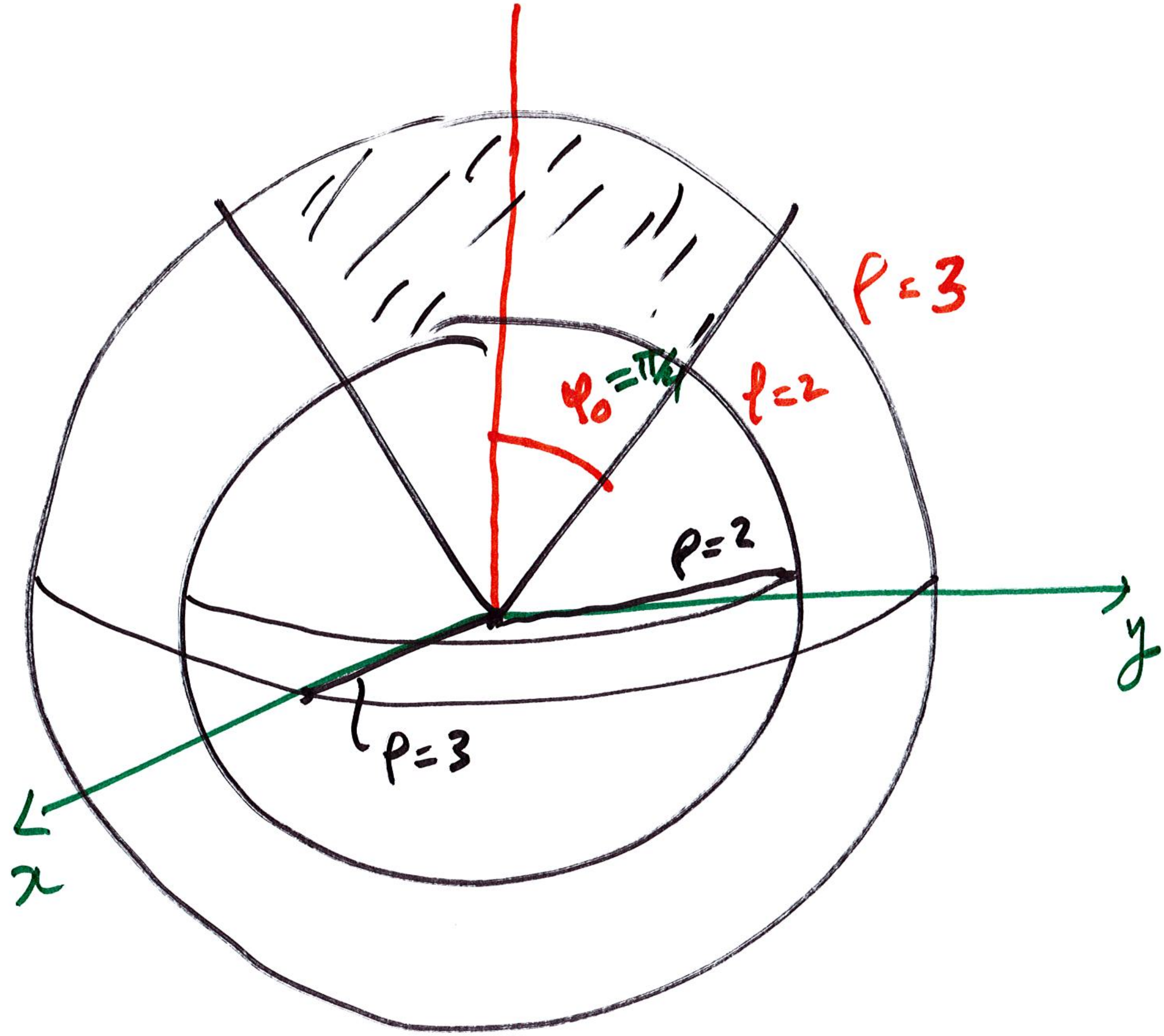


Example: Compute the integral $\iiint_E x \, dV$

$E$ is the region of the 1st octant bounded by the spheres $x^2+y^2+z^2=4$ $x^2+y^2+z^2=9$ and the cone $z=\sqrt{x^2+y^2}$.

$z=\rho\cos\varphi$; $\sqrt{x^2+y^2}=\rho\sin\varphi$.

$\rho\cos\varphi=\rho\sin\varphi$; $\cos\varphi=\sin\varphi$.




$0 \leq \varphi \leq \pi/4$.

$0 \leq \theta \leq 2\pi$

$E$: $2 \leq \rho \leq 3$, $0 \leq \theta \leq \pi/2$.

$0 \leq \varphi \leq \pi/4$.

$$\iiint_E f\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\underbrace{\overbrace{\rho\sin\varphi\cos\theta}^{x}}, \underbrace{\overbrace{\rho\sin\varphi\sin\theta}^{y}, \overbrace{\rho\cos\varphi}^{z}})\, \underbrace{\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi}$$

$$E = \quad a \le \rho \le b, \quad \alpha \le \theta \le \beta$$

$$c \le \varphi \le d$$

$$\iiint_E \underline{\underline{x}}\, dV =$$

$$= \int_0^{\pi/4}\int_0^{\pi/2}\int_2^3 \underline{\underline{\rho}} \sin\varphi \cos\theta \cdot \underline{\underline{\rho^2}} \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$\int_2^3 \rho^3 \, d\rho = \frac{1}{4}\rho^4 \Big|_2^3 = \frac{1}{4}(81-16)$$

$$= 65/4$$

$$= \frac{65}{4}\int_0^{\pi/4}\int_0^{\pi/2} \sin^2\varphi \cos\theta \, d\theta \, d\varphi. \qquad \int_0^{\pi/2} \cos\theta \, d\theta = 1.$$

$$= \frac{65}{4}\int_0^{\pi/4} \sin^2\varphi \, d\varphi = \frac{65}{4}\left(\frac{\pi}{8} - \frac{1}{4}\right).$$

$$\sin^2\varphi = \frac{1-\cos 2\varphi}{2}.$$

$$\int_0^{\pi/4} \sin^2\varphi \, d\varphi = \int_0^{\pi/4} \left( \frac{1-\cos 2\varphi}{2} \right) d\varphi$$

$$= \pi/8 - \frac{1}{4}$$

$$\int_0^{\pi/4} \cos 2\varphi \, d\varphi = \frac{1}{2} \sin 2\varphi \Big|_0^{\pi/4} = \frac{1}{2}$$