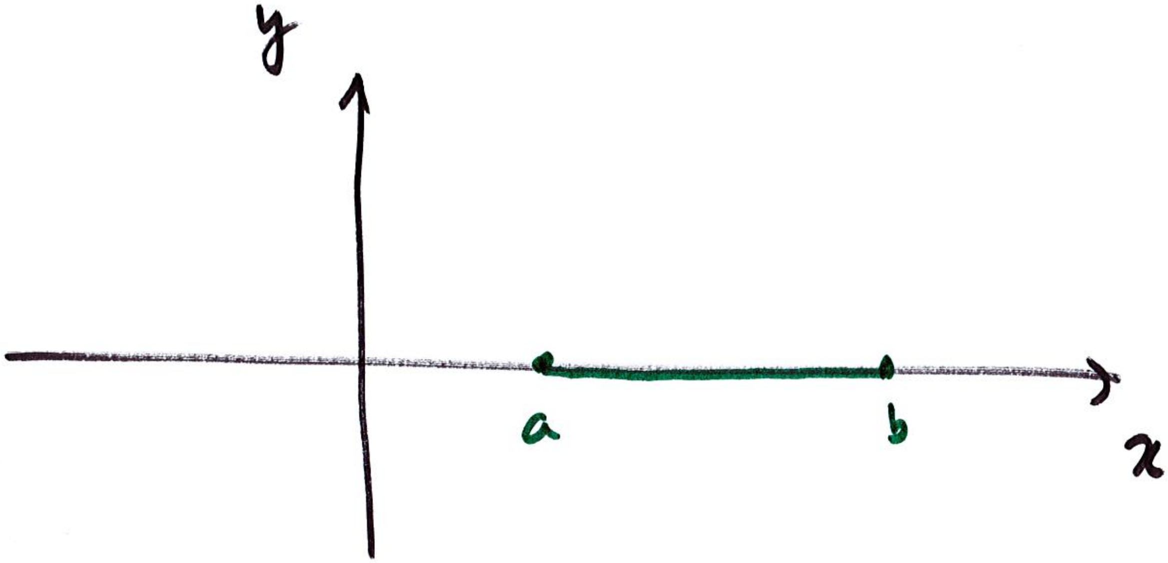
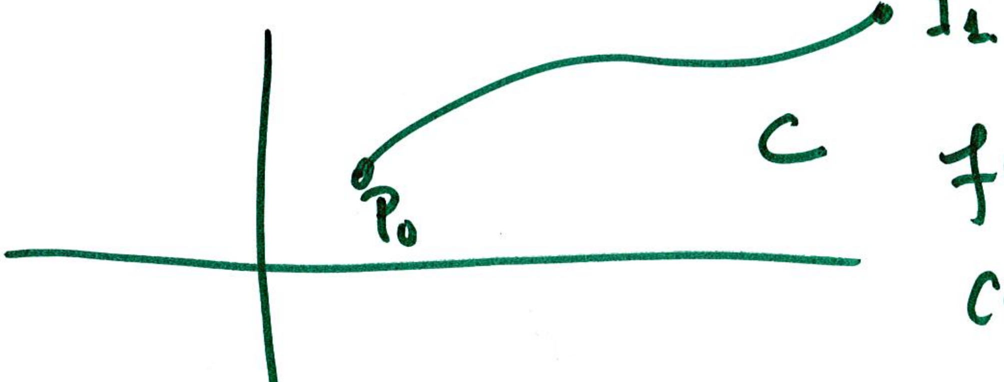
Lesson 27 Section 16.2

Line Integrals

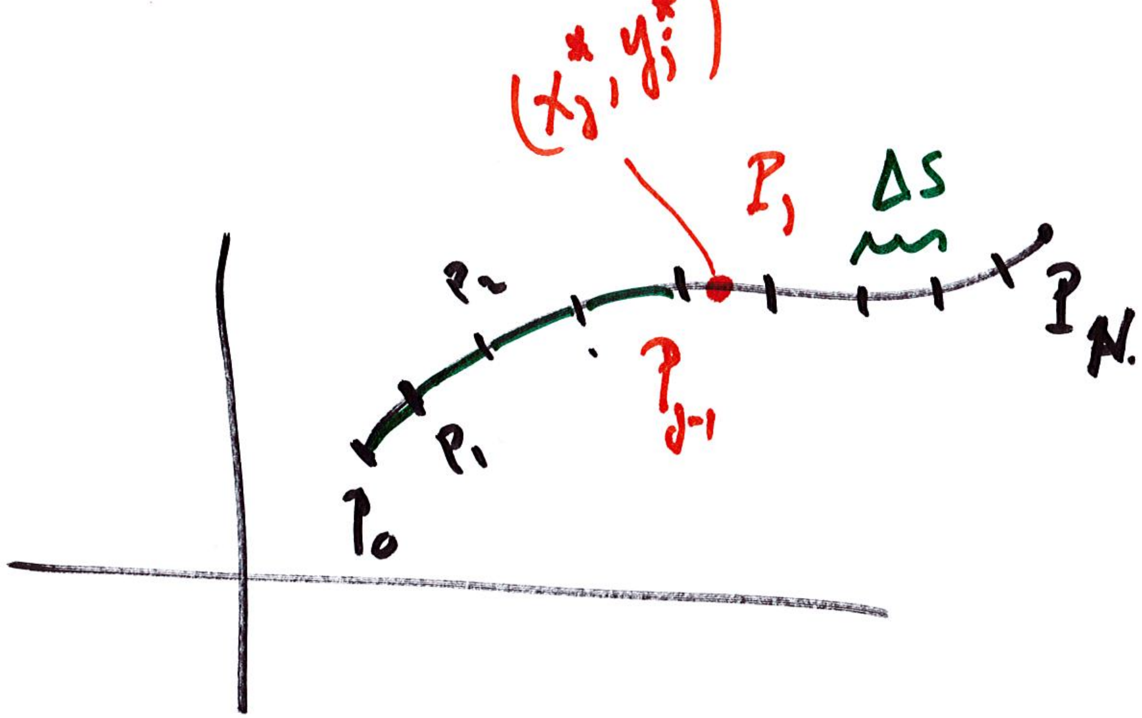


$$\int_a^b f(x)\,dx$$



$f(x,y)$
continuous

$$\int_C f(x,y)\, ds \qquad s = \text{arc-length parameter.}$$



I. Step 1: Divide the curve into N equal parts

II Step 2 In each segment of the curve

P_{j-1} to P_j

pick a point (x_j^*, y_j^*)

III. Step 3

$$\sum_{j=1}^{N} f(x_j^*, y_j^*)\,\Delta s$$

Step 4:

$$\int_C f(x,y)\,ds = \lim_{N\to\infty} \sum_{j=1}^{N} f(x_j^*, y_j^*)\,\Delta s$$

How do we compute such an integral?

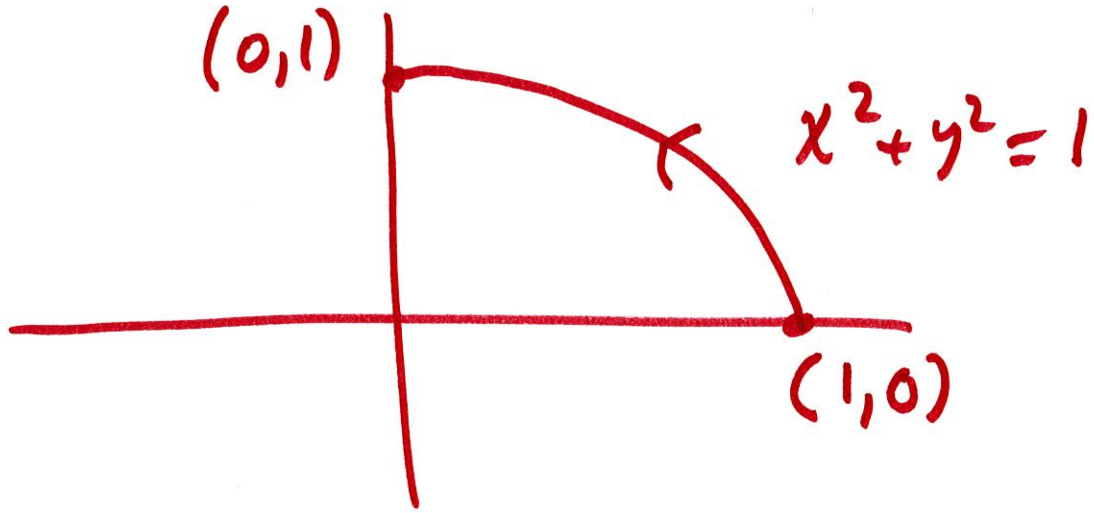
We parametrize the curve C

$$x = x(t)\ ,\quad y = y(t) \quad a \le t \le b$$

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

$$\boxed{\int_C f(x,y)\, ds = \int_a^b f(x(t); y(t)) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt}$$

Example:
$$\int_C x\, y\, ds$$



Step 1: Parametrize the curve.

$$x = \cos t \quad , \quad y = \sin t \qquad 0 \leq t \leq \pi/2$$

$$x' = -\sin t \quad , \quad y' = \cos t.$$

Step 2:

$$\int \underset{\cos t}{\underset{\|}{x}}\ \underset{\sin t}{\underset{\|}{y}}\ ds = \int_0^{\pi/2} \cos t \cdot \sin t \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$$

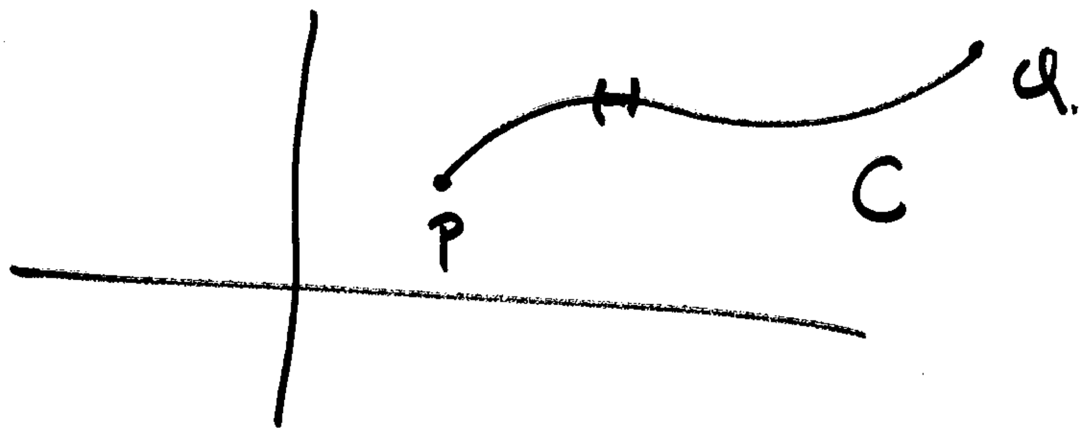
$$\int_C xy\,ds = \int_0^{\pi/2} \underbrace{\sin t}_{u}\ \underbrace{\cos t\ dt}_{du}$$

$u = \sin t, \quad du = \cos t\,dt$

$$= \int_0^1 u\,du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}.$$

$$\int_C xy\,ds = \frac{1}{2}.$$

Application:



C = thin wire

density of mass = $\rho(x,y)$.
per unit of length

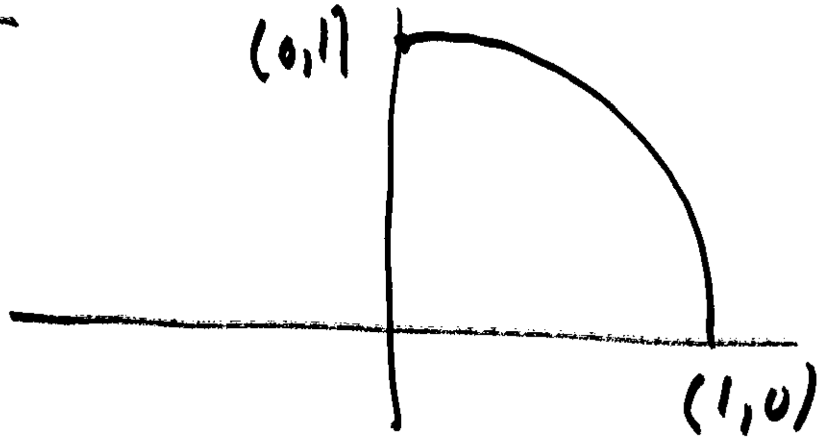
Total mass = $\int_C \rho(x,y)\, ds = m$

Center of mass of the wire.

$$\bar{x} = \frac{1}{m}\int_C x\,\rho(x,y)\, ds$$

$$\bar{y} = \frac{1}{m}\int_C y\,\rho(x,y)\, ds.$$

Example:



$\rho(x,y) = x^2 y.$

Find its center of mass

(1) $m = \int_C \rho(x,y)\, ds$

$x = \cos t, \quad y = \sin t.$

$ds = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

$= dt.$

$$m = \int x^2 y \, ds =$$

$$= \int_0^{\pi/2} \underbrace{\cos^2 t}_{u^2} \cdot \underbrace{\sin t \, dt}_{-du}$$

$u = \cos t$, $du = -\sin t \, dt$

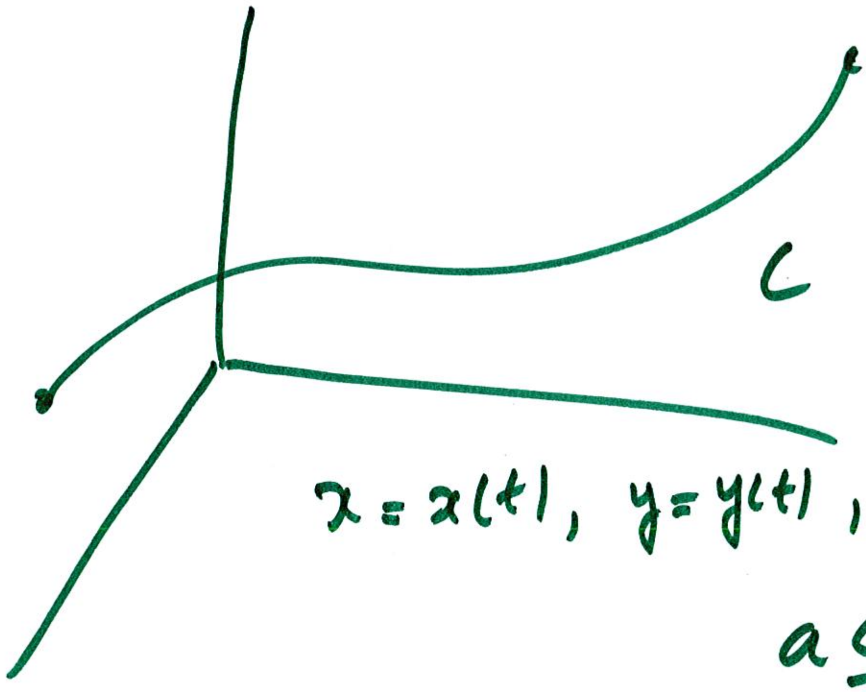
$$= \int_1^0 -u^2 \, du = \int_0^1 u^2 \, du = 1/3 .$$

$$\bar{x} = \frac{1}{m} \int x \, \rho(x,y) \, ds$$

$$= \frac{1}{m} \int x^3 y \, ds = \frac{1}{m} \int \cos^3 t \cdot \sin t \, dt$$

$$= \frac{1}{m} \cdot 1/4 = 3/4 .$$

Exercise: Compute $\bar{y}$.



$x = x(t), \quad y = y(t), \quad z = z(t)$

$a \leq t \leq b.$

$$\int_C f(x,y,z)\, ds =$$

$$= \int_a^b f(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

Example: Compute

$$\int_C (x^2+y^2+z^2)\, ds$$

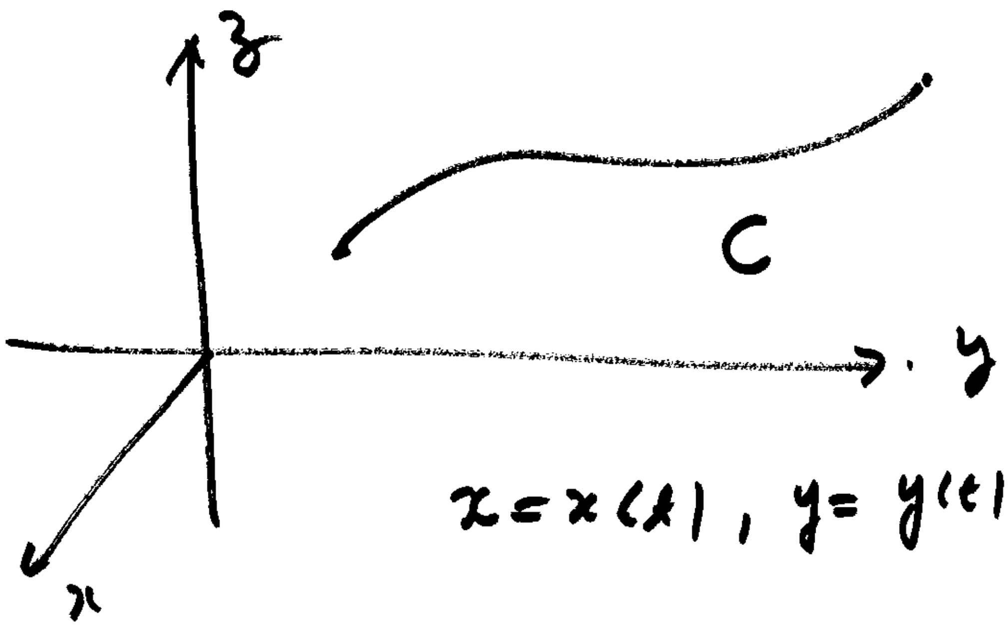
$x = t, \quad y = 2\cos t, \quad z = 2\sin t$

$0 \le t \le 2\pi$

$$\int_C (x^2+y^2+z^2)\, ds =$$

$$= \int_0^{2\pi} (t^2 + 4\cos^2 t + 4\sin^2 t) \cdot \sqrt{1 + (-2\sin t)^2 + (2\cos t)^2}\, dt$$

$$= \int_0^{2\pi} (t^2+4) \cdot \sqrt{5}\, dt = \sqrt{5} \int_0^{2\pi} (t^2+4)\, dt.$$



$x = x(t)$, $y = y(t)$, $z = z(t$

$f(x,y,z)$ continuous $\quad x = x(t)$

$dx = x'(t)\,dt$

$$\int_C f(x,y,z)\,\underline{dx} = \int_a^b f(x(t), y(t), z(t))\, x'(t)\,dt.$$

$$\int_C f(x,y,z)\,dy = \int_a^b f(x(t), y(t), z(t))\, y'(t)\,dt$$

$$\int_C f(x,y,z)\,dz = \int_a^b f(x(t), y(t), z(t))\, z'(t)\,dt.$$

Example: Compute

$$\int_C y\,dx + z\,dy + x\,dz$$

$y = t, \; dy = dt$

$C: \; x = \sqrt{t}, \; \underbrace{y = t}, \; z = t^2$

$x = \sqrt{t} = t^{1/2}$

$dx = \frac{1}{2} t^{-1/2} dt$

$1 \le t \le 4.$

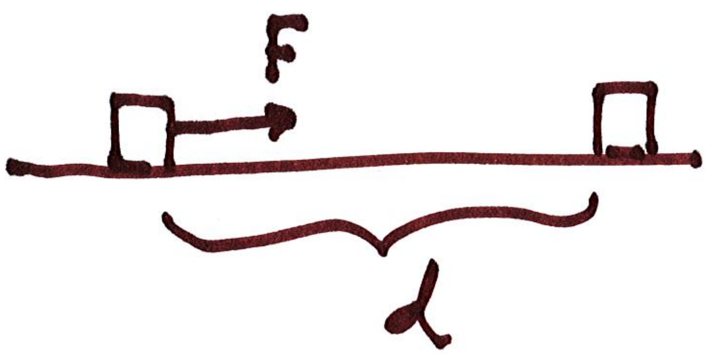
$$\int_C \underbrace{y}_{t} \, \underbrace{dx} + \underbrace{z}_{t^2} \, dy + x\,dz =$$

$$= \int_1^4 t \cdot \frac{1}{2} t^{-1/2} \underline{dt} + t^2 \underline{dt} + t^{1/2} \, 2t \, \underline{dt}$$

$$= \int_1^4 \left(\frac{1}{2} t^{1/2} + t^2 + 2t^{3/2} \right) dt.$$

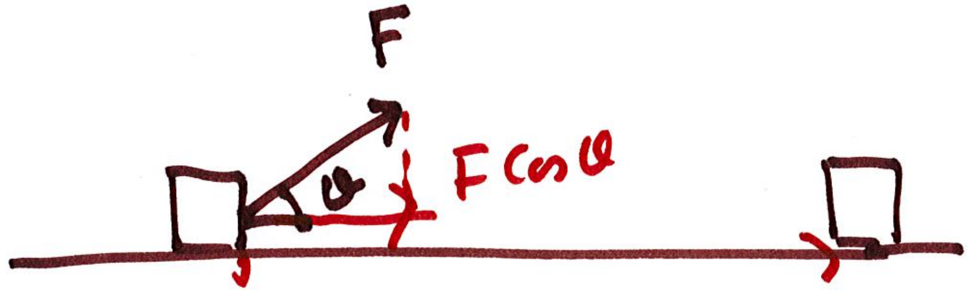
Line Integrals of Vector Fields

Work done by a Force

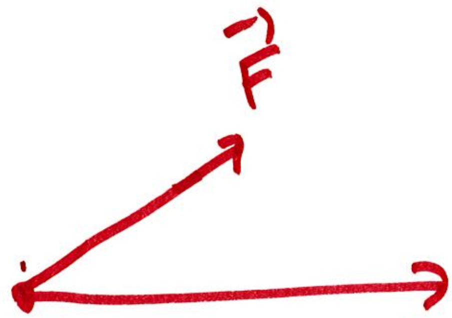


F = constant

Work = $F \cdot d$



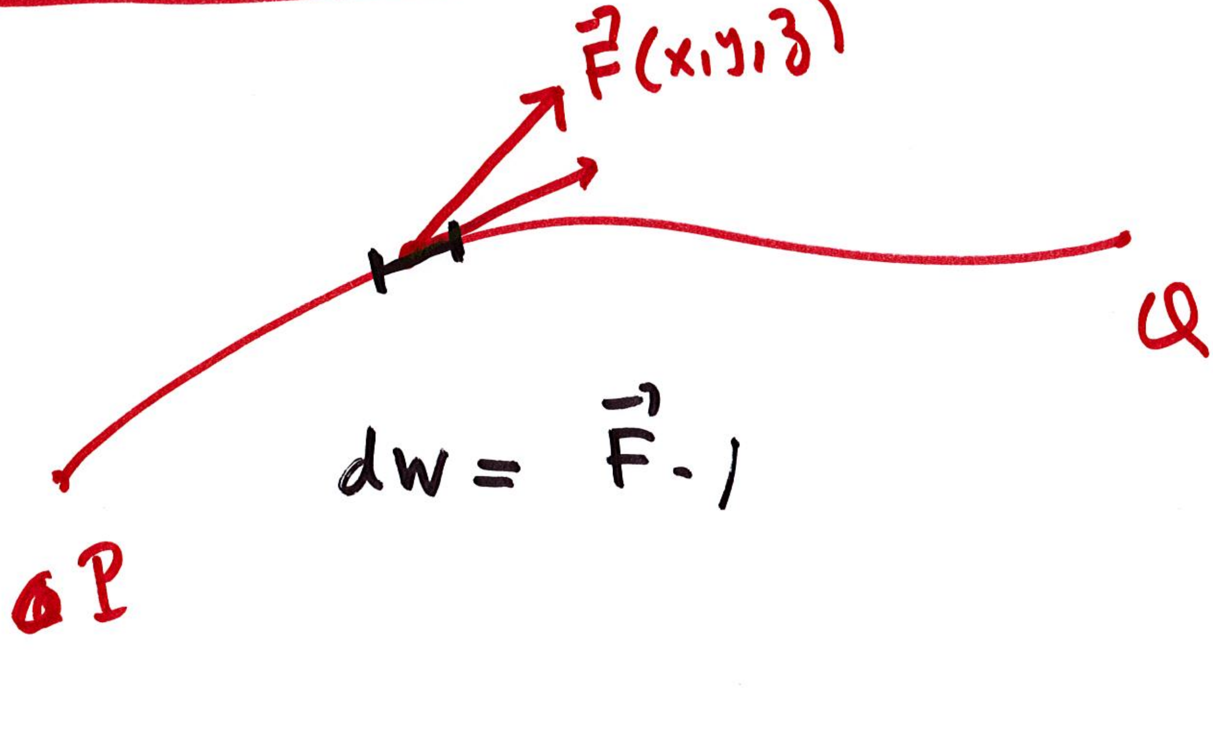
Work = $F \cdot d \cos\theta$.



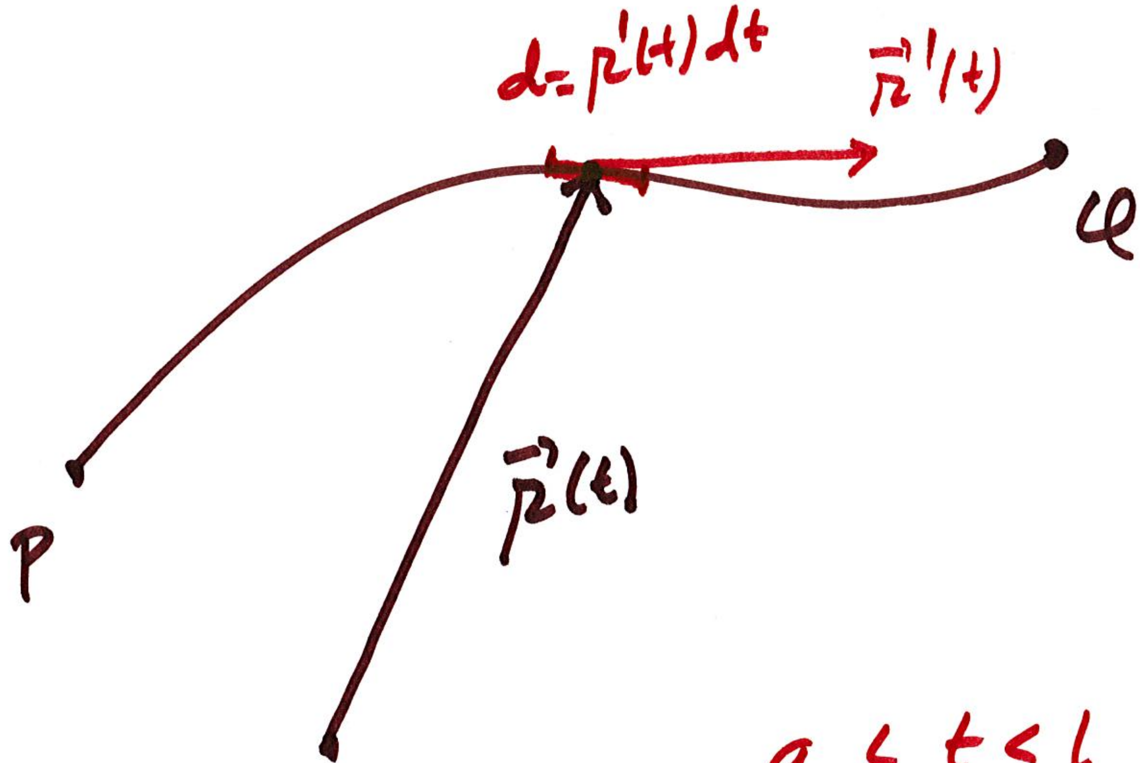
d = displacement vector.

$$W = \underline{\underline{\vec{F}}} \cdot \underline{\underline{\vec{d}}}.$$

Non-constant force



$$dw = \vec{F} \cdot l$$



$a \le t \le b$

$$C: \quad x = x(t), \quad y = y(t), \quad z = z(t)$$

$$\vec{r} = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}.$$

$$dW = \vec{F} \cdot \vec{r}'(t)\,dt$$

$$\text{Total work} = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{r}'(t)\,dt$$