Exam 2

April 3

6:30 PM

Elliott Hall

Lessons 18 to 29

Lesson 28 Continuation of

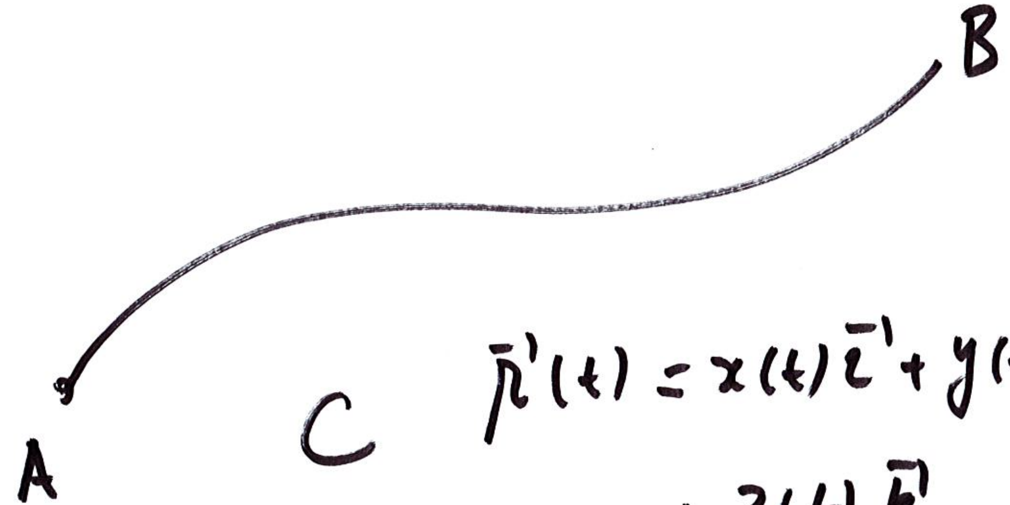
Section 16.2.

Line Integrals of Vector

Fields.

Work.

$$\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}.$$



C $\quad \vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}.$

$$a \le t \le b$$

How much work (energy) does it take to move an object from A to B along C.

$$\text{Work} = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \frac{d}{dt}\vec{r}(t)\, dt$$

This is the integral of the vector field $\vec{F}$ along C.

<u>Remark:</u> This definition is independent of the choice of the parametrization.

In other words if C can be ~~cont~~ parametrized by

$$x = \tilde{x}(t), \quad y = \tilde{y}(t), \quad z = \tilde{z}(t)$$

$$\vec{\tilde{r}} = \tilde{x}(t)\vec{i} + \tilde{y}(t)\vec{j} + \tilde{z}(t)\vec{k}.$$

$$c \le t \le d.$$

$$\int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \frac{d}{dt}\vec{r}(t)\, dt$$

$$= \int_c^d \vec{F}(\tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \cdot \frac{d}{dt}\vec{\tilde{r}}(t)\, dt.$$

Notation:

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \underbrace{F(x(t), y(t), z(t))} \cdot \underbrace{\frac{d}{dt}}\vec{r}\, dt.$$

Examples: Compute $\int_C \vec{F} \cdot d\vec{r}$

where C is given by $\vec{r}(t)$.

(1) $F(x,y,z) = (x+y^2)\vec{i} + xz\vec{j} + (y+z)\vec{k}$

$\vec{r}(t) = t^2\vec{i} + t^3\vec{j} - 2t\,k \quad 0 \le t \le 2.$

$= x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

$x(t) = t^2$

$y(t) = t^3$

$z(t) = -2t.$

Step 1: $F(x(t), y(t), z(t)) = .$

$F(x(t), y(t), z(t)) = (t^2 + t^6)\vec{i} + (-2t^3)\vec{j} + (t^3 - 2t)\vec{k}.$

Step 2

$\frac{d\vec{r}}{dt} = 2t\vec{i} + 3t^2\vec{j} - 2\vec{k}.$

Step 3: $\vec{F}(x(t), y(t), z(t)) \cdot \frac{d\vec{r}}{dt}$.

$$= 2t(\cancel{t^2} + t^6) - 6t^5 + \cancel{2t^3} - 2\cancel{t^3} + 4t$$

$$= \cancel{2t^3} + 2t^7 - 6t^5 + 4t$$

Step 4 $\displaystyle\int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \frac{d\vec{r}}{dt}$

$$= \int_0^2 (4t + 2t^7 - 6t^5)\, dt$$

$$= 2t^2 + t^8/4 - t^6 \Big|_0^2$$

$$= 8 + \cancel{\frac{2^8}{2^2}} - \cancel{2^6} = 8.$$

#32) Find the work done by the force field

$$\vec{F}(x,y) = x^2\,\vec{i} + 2xy\,\vec{j}$$

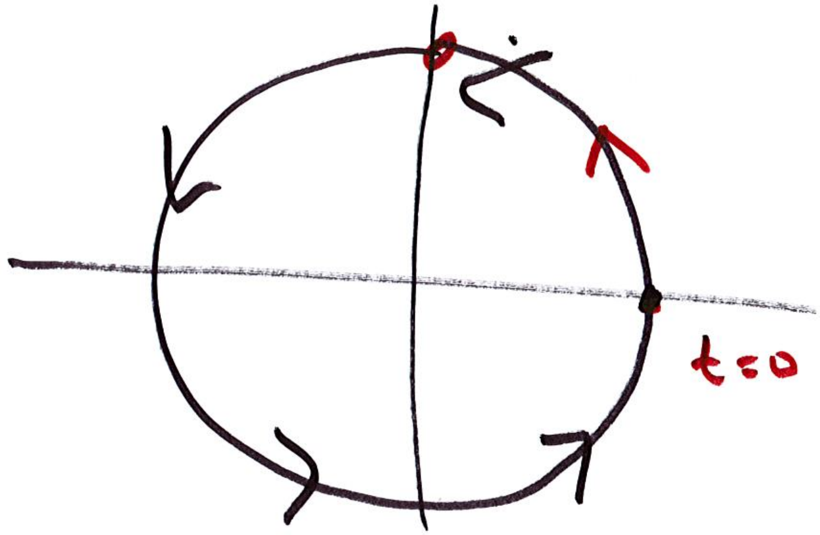
on a particle that moves once around the circle

$$x^2 + y^2 = 4$$

in the counterclockwise direction.

Solution: $W = \int_C \vec{F} \cdot d\vec{r}$.

First we parametrize the curve.



$x(t) = 2\cos t \quad , \quad y = 2\sin t.$

$x(0) = 2 \quad , \quad y(0) = 0$

$t = \pi/2 \quad x(\frac{\pi}{2}) = 2.\cos\pi/2 = 0, \quad y(\pi/2) = 2.$

$\cdot \ \vec{F}(x(t), y(t)) = 4\cos^2 t\ \vec{i} + 8\cos t \sin t\ \vec{j}.$

$\frac{d\vec{r}}{dt} = -2\sin t\ \vec{i} + 2\cos t\ \vec{j}.$

$$\vec{F} \cdot \frac{d\vec{r}}{dt} = -8 \sin t \cos^2 t$$

$$+ 16 \cos^2 t \sin t$$

$$\boxed{\vec{F} \cdot \frac{d\vec{r}}{dt} = 8 \cos^2 t \sin t}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} 8 \cos^2 t \sin t \, dt$$

$\cos t = u \quad , \quad du = -\sin t \, dt$

$$\int \cos^2 t \sin t \, dt = \int -u^2 \, du = -\frac{u^3}{3} = -\frac{\cos^3 t}{3}$$

$$\int_0^{2\pi} 8 \cos^2 t \sin t \, dt = -\frac{8}{3} \cos^3 t \Big|_0^{2\pi} = 0$$

$$\vec{F} = xy\,\vec{i} + z\,\vec{j} - yx^2\,\vec{k}$$

C is the segment from $\underline{\underline{(1,1,0)}}$ to $(2,1,1)$

$$\int_C \vec{F}\cdot d\vec{r}.$$

<u>Step 1</u>: Parametrize C.

$$C: \langle 1,1,0\rangle + t\langle 1,0,1\rangle.$$

$x(t) = 1+t, \quad y(t) = 1, \quad z(t) = t.$

$\bullet \; \vec{r}(t) = (1+t)\,\vec{i} + \vec{j} + t\,\vec{k}$

$\frac{d\vec{r}}{dt} = \vec{i} + \vec{k}$

$\vec{F}(x(t), y(t), z(t)) = (1+t)\,\vec{i} + t\,\vec{j} - (1+t)^2\,\vec{k}$

$\bullet$

$\vec{F} \cdot \frac{d\vec{r}}{dt} = (1+t) - (1+t)^2.$

$$\int_C \vec{F} \cdot dr = \int_0^1 \left[(1+t) - (1+t)^2 \right] dt.$$

$1+t = u$

$$= \int_1^2 (u - u^2)\, du = \frac{u^2}{2} - \frac{u^3}{3} \Big|_1^2.$$