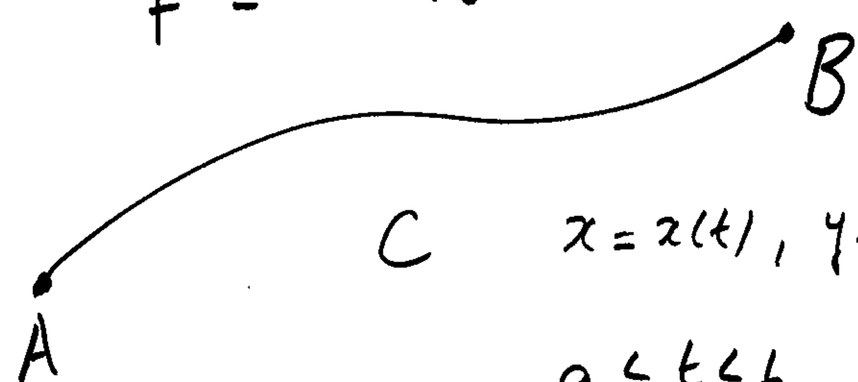
Lesson 29 Section 16.3

The Fundamental theorem of Line Integrals.

$$\vec{F} = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$$



$x = x(t),\ y = y(t),\ z = z(t)$

$a \le t \le b$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{r}\,'(t)\,dt$$

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$$

$$\vec{r}\,'(t) = x'(t)\,\vec{i} + y'(t)\,\vec{j} + z'(t)\,\vec{k}.$$

Special Case

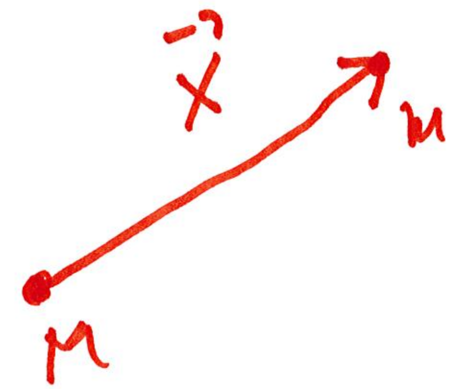
$$\vec{F}(x,y,z) = \vec{\nabla} f(x,y,z) =$$

$$\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}.$$

In this case we say that $\vec{F}$ is a conservative vector field.

Examples:

(1) Gravitational Force.



$$\vec{F} = -\frac{G\,m\,M}{|\vec{x}|^2}\,\frac{\vec{x}}{|\vec{x}|}$$

$$\vec{F} = -\frac{\overbrace{G\,m\,M}^{K}}{|\vec{x}|^3}\,\vec{x}.$$

$$f(x,y,z) = \frac{K}{|\vec{x}|} \qquad \vec{x} = (x,y,z)$$

$$|\vec{x}| = (x^2+y^2+z^2)^{1/2}$$

$K = m M G.$

$$f(x,y,z) = K\,(x^2+y^2+z^2)^{-1/2}.$$

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$\frac{\partial f}{\partial x} = K\left(-\tfrac{1}{2}\right).\,(x^2+y^2+z^2)^{-3/2}.\,2x$$

$$= -\,K\,x\,(x^2+y^2+z^2)^{-3/2}$$

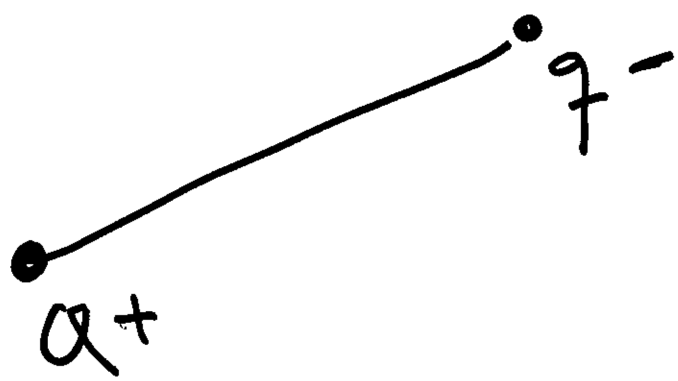
$$= -\,K\,\frac{x}{|\vec{x}|^3}$$

$$\frac{\partial f}{\partial y} = -K\frac{y}{|\vec{x}|^3}\ ; \quad \frac{\partial f}{\partial z} = -\,K\,\frac{z}{|\vec{x}|^3}.$$

$$\vec{\nabla} f = K\left(\frac{-x}{|\vec{x}|^3}\vec{i} - \frac{y}{|\vec{x}|^3}\vec{j} - \frac{z}{|\vec{x}|^3}\vec{k}\right)$$

$$= -\frac{K}{|\vec{x}|^3}\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$\boxed{\vec{\nabla} f = -\frac{K}{|\vec{x}|^3}\vec{x} = F.}$$



$$\vec{F} = -M\frac{Qq}{|\vec{x}|^3}\vec{x}.$$

Claim:

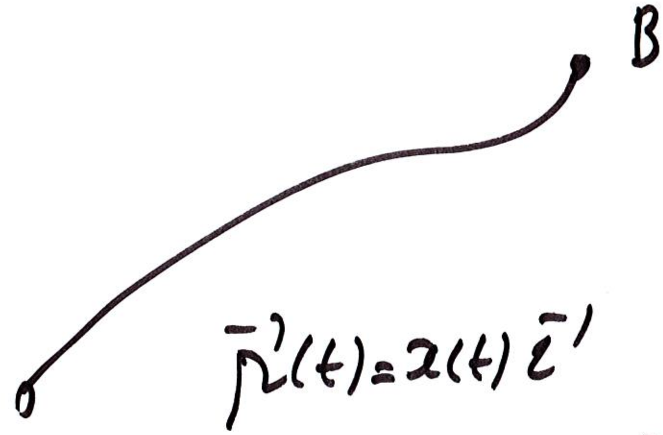
$$\frac{\partial f}{\partial x}(x(t), y(t), z(t))\,\frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t), y(t), z(t))\,\frac{dy}{dt}$$

$$+ \frac{\partial f}{\partial z}(x(t), y(t), z(t))\,\frac{dz}{dt} =$$

$$\vec{\nabla} f\,(x(t), y(t), z(t)) \cdot \overbrace{\left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}\right)}^{d\vec{r}/dt}$$

$$= \frac{d}{dt} f\,(x(t), y(t), z(t)).$$

$$\int_C \vec{\nabla} f \cdot d\vec{r}$$



$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}.$$

$$a \le t \le b.$$

$$\int_C \vec{\nabla} f \cdot d\vec{r} = \int_a^b \vec{\nabla} f\,(x(t), y(t), z(t)) \cdot \frac{d\vec{r}}{dt}\,dt$$

The Chain Rule

$$\frac{d}{dt} f\,(x(t), y(t), z(t)) = \frac{\partial f}{\partial x}(x(t), y(t), z(t)) \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t), y(t), z(t))\,\frac{dy}{dt} + \frac{\partial f}{\partial z}(x(t), y(t), z(t))\,\frac{dz}{dt}.$$

$$\int_C \vec{\nabla} f \cdot d\vec{r} =$$

$$= \int_a^b \vec{\nabla} f(x(t), y(t), z(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$

$$= \int_a^b \frac{d}{dt} f(x(t), y(t), z(t))\, dt.$$

$$= f(x(b), y(b), z(b)) -$$

$$f(x(a), y(a), z(a)).$$

$$= f(B) - f(A).$$

Example: $f(x,y,z) = xyz + 3z^2 + y^2$

$$\int_C \vec{\nabla} f \cdot dr = f(0, e^{\pi}, -e^{\pi}) - f(0,1,1).$$

where C is the curve parametrized by

$x(t) = \sin t \, \cancel{\ } \cos t$

$y(t) = e^t$

$z(t) = e^t \cos t \qquad 0 \leq t \leq \pi.$

__$t=0$__: $x(0) = 0$, $y(0) = 1$, $z(0) = 1$

$A = (0, 1, 1)$

$t = \pi$: $x(\pi) = 0$, $y = e^{\pi}$, $z = -e^{\pi}$.

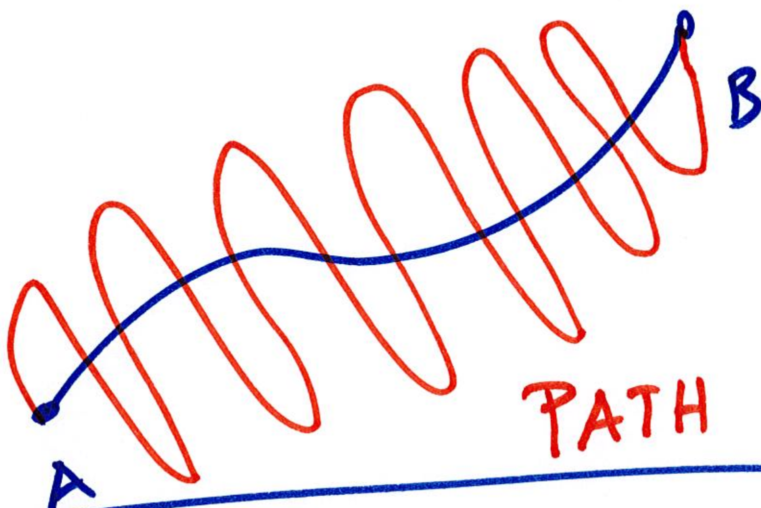
$B = (0, e^{\pi}, -e^{\pi})$.

Conclusions If $\vec{F} = \vec{\nabla} f$

I) $$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

B = end point of C

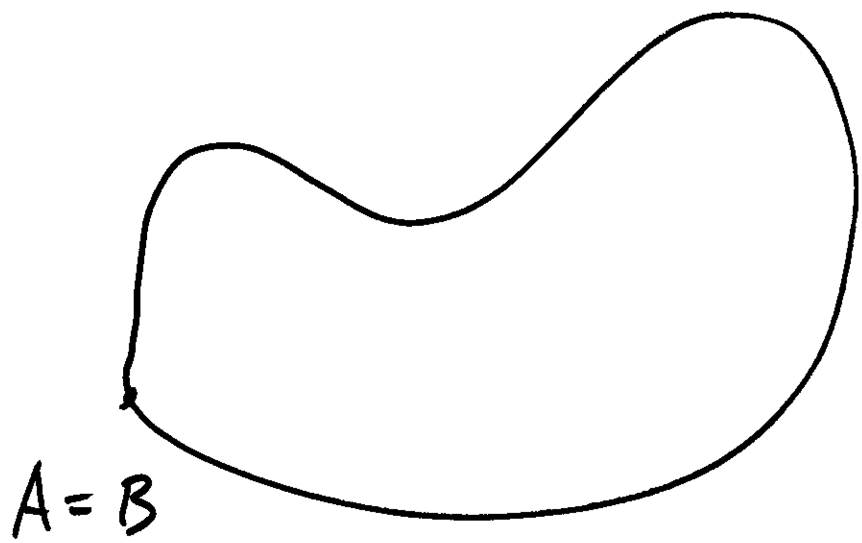
A = starting point of C.



PATH INDEPENDENCE.

The integral does not depend on the curve; it only depends on the end points

II) If C is a closed curve



Starting point = End Point

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

Problem: $F(x,y,z) = P\vec{i} + Q\vec{j} + R\vec{k}$

Is $\vec{F} = \vec{\nabla} f$?

Test 1 (necessary condition)

If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$$= \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$P = \frac{\partial f}{\partial x} \; ; \quad Q = \frac{\partial f}{\partial y} \; , \quad R = \frac{\partial f}{\partial z} .$$

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \; ; \quad \frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

If the second partial derivatives of f are continuous.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

$$\frac{\partial P}{\partial z} = \frac{\partial^2 f}{\partial z \partial x}, \quad \frac{\partial R}{\partial x} = \frac{\partial^2 f}{\partial x \partial z}$$

$$\boxed{\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}}$$

$$\frac{\partial Q}{\partial z} = \frac{\partial^2 f}{\partial z \partial y}, \quad \frac{\partial R}{\partial y} = \frac{\partial^2 f}{\partial y \partial z}$$

$$\boxed{\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}}$$

In 2 dimensions $\underline{R=0}$

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

<u>Question:</u> Is

$$\vec{F} = xy\,\vec{i} + zyx\,\vec{j} + z^3\,\vec{k}$$

Conservative.?

$P = xy,\quad Q = zyx\ ,\ R = z^3.$

$$\frac{\partial P}{\partial y} = x,\quad \frac{\partial Q}{\partial x} = zy$$

Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ $\underline{\underline{\text{No}}}$ F is not conservative.

$$\vec{F} = (2xyz + 3y\cos(xy))\,\vec{i} + (x^2 z + 3x\cos(xy))\,\vec{j} + x^2 y\,\vec{k}.$$

$$P = 2xyz + 3y\cos(xy)$$

$$Q = x^2 z + 3x\cos(xy)$$

$$R = x^2 y.$$

Check that $\vec{F}$ passes the 3 tests:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$$

$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

$$P = \frac{\partial f}{\partial x}, \quad Q = \frac{\partial f}{\partial y}, \quad R = \frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 2xyz + 3y\cos(xy)$$

$$f(x,y,z) = \int [2xyz + 3y\cos(xy)]\,dx + C(y,z)$$

$$= x^2yz + 3\sin(xy) + \underline{\underline{C(y,z)}}.$$

$$\frac{\partial f}{\partial y} = \underline{x^2z} + \underline{3x\cos}(xy) + \frac{\partial C}{\partial y}$$

$$= Q = \underline{x^2z + 3x\cos(xy)}$$

$$\frac{\partial C}{\partial y} = 0 \qquad C = C(z)$$

$$\frac{\partial f}{\partial z} = R = x^2 y$$

$$\frac{\partial f}{\partial z} = x^2 y + \frac{\partial c}{\partial z} = x^2 y$$

$$\frac{\partial c}{\partial z} = 0$$

$$C = \text{Constant}$$

$$\boxed{f(x, y, z) = x^2 z y + 3 \sin(xy) + c}$$