Lesson 30 Section 16.4

Green's theorem

Reminder: Exam 2 4/3

6:30 in Elliott

Covers lessons 18 - 29

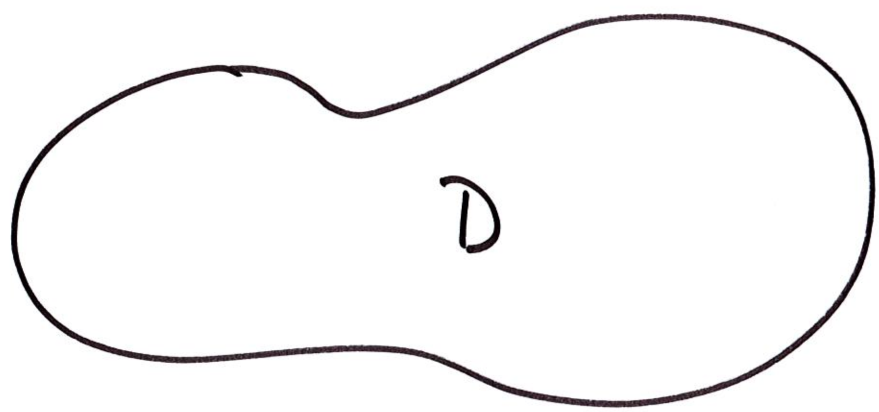
Monday: Review for exam 2

No Class on Wed 04/04
because of the exam.

Green's theorem is a form
of ~~for~~ the fundamental theorem

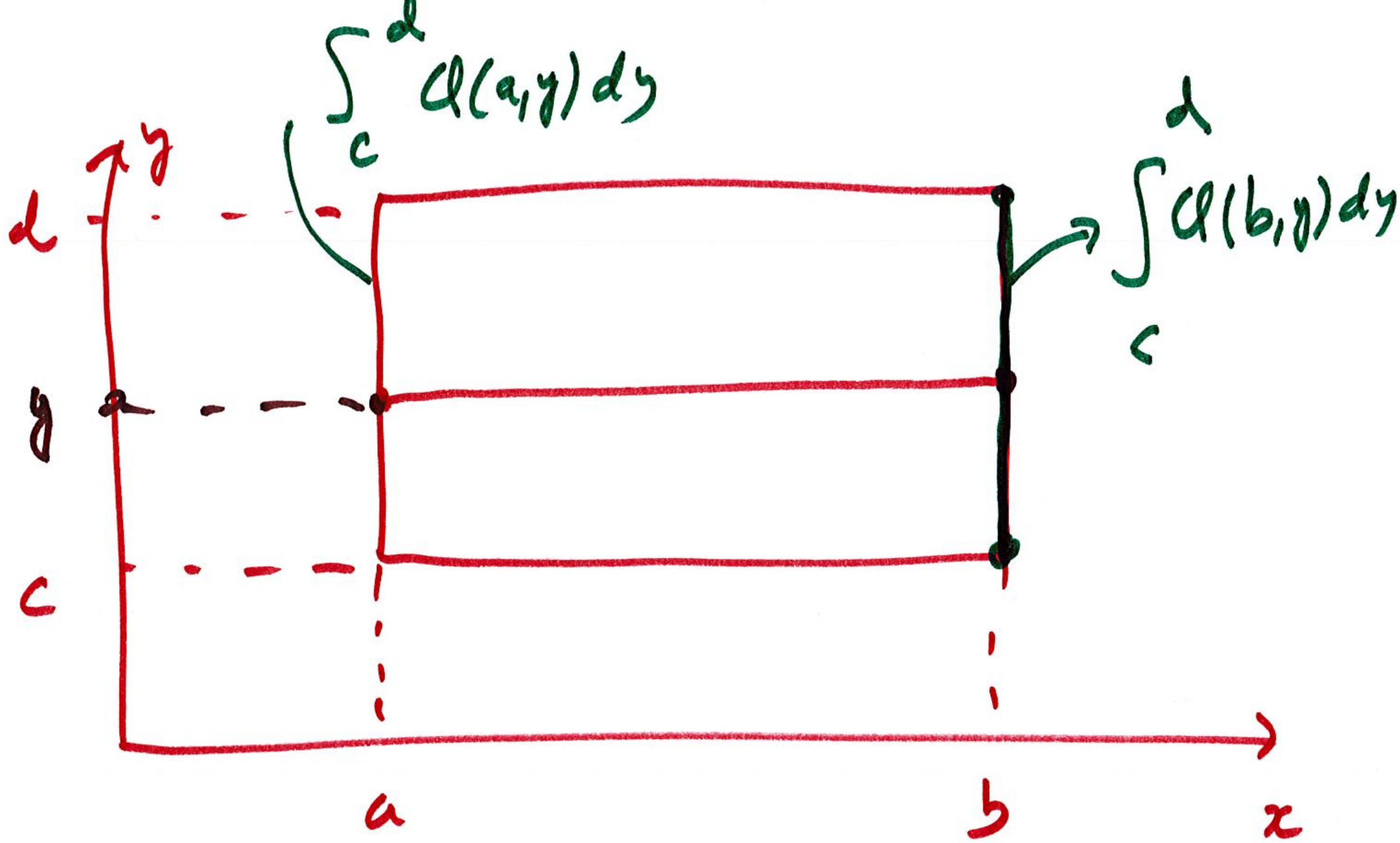
of Calculus for functions of 2 variables

$$\iint_D \frac{\partial Q}{\partial x} \, dA.$$



$$\iint_D \frac{\partial Q}{\partial x} \, dA = \iint_D \frac{\partial Q}{\partial x} \, dx \, dy$$

If D is a rectangle



$$D = [a,b] \times [c,d]$$

$$\iint_D \frac{\partial Q}{\partial x}\, dA = \int_c^d \Big(\underbrace{\int_a^b \frac{\partial Q}{\partial x}\, dx}\ \Big)\, dy$$

The fundamental theorem of Calculus (F. T. C)

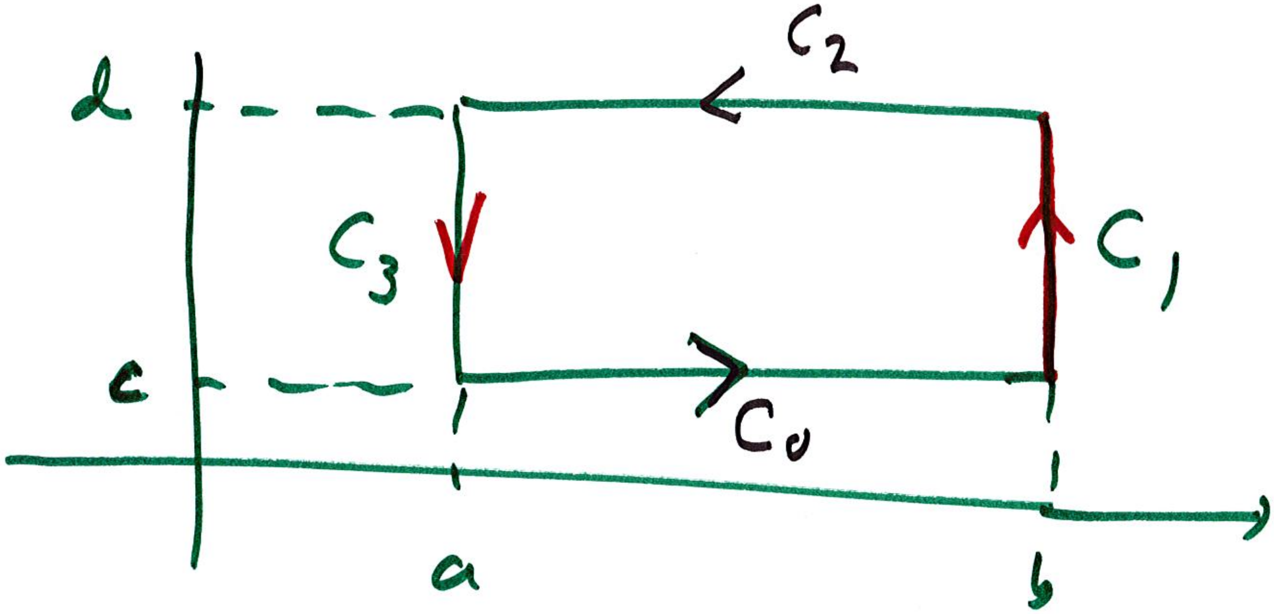
$$\int_a^b f'(t)\,dt = f(b) - f(a).$$

$$\int_a^b \frac{\partial Q}{\partial x}(x, y)\,dx = Q(b, y) - Q(a, y)$$

$$\uparrow$$

We apply the F. T. C in the x variable.

$$\iint_D \frac{\partial Q}{\partial x}\,dA = \int_c^d \left(Q(b, y) - Q(a, y) \right) dy.$$



$$\int_c^d Q(b,y)\,dy - \int_c^d Q(a,y)\,dy$$

$$= \int_c^d Q(b,y)\,dy + \int_d^c Q(a,y)\,dy$$

$$\int_{C_0} Q\,dy = 0 \qquad \int_{C_2} Q\,dy = 0$$

Conclusion:

$$\iint_D \frac{\partial Q}{\partial x}\, dA = \int_C Q\, dy.$$

$$C = C_0 \cup C_1 \cup C_2 \cup C_3$$

$$\iint_D \frac{\partial Q}{\partial x}\, dA = \int_{C_1} Q\, dy + \int_{C_3} Q\, dy$$

$$= \int_C Q\, dy$$

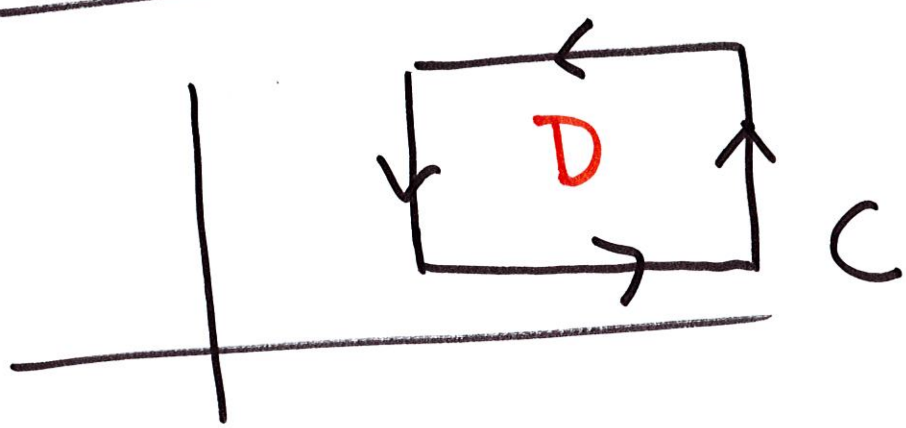
~~because~~ because $\int_{C_0} Q\, dy = \int_{C_2} Q\, dy = 0$

If D is a rectangle

$$[a, b] \times [c, d]$$

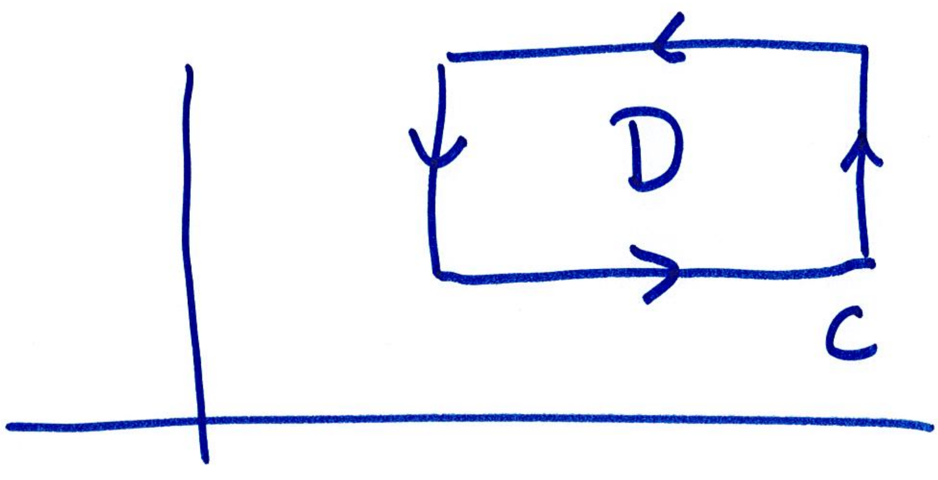
$$\iint_D \frac{\partial Q}{\partial x}\, dA = \int_C Q\, dy$$

C is the <u>border</u> of the rectangle oriented <u>Counterclockwise.</u>



$$\iint_D -\frac{\partial P}{\partial y}\, dA = \int_C P\, dx.$$

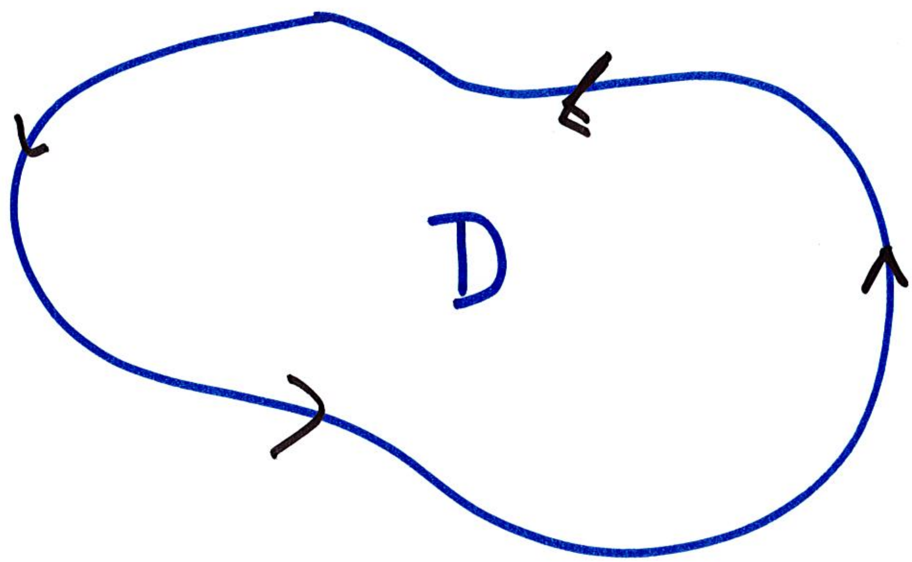
Green's theorem for a rectangle:



$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \int_C P\, dx + Q\, dy.$$

Green's theorem in full generality.

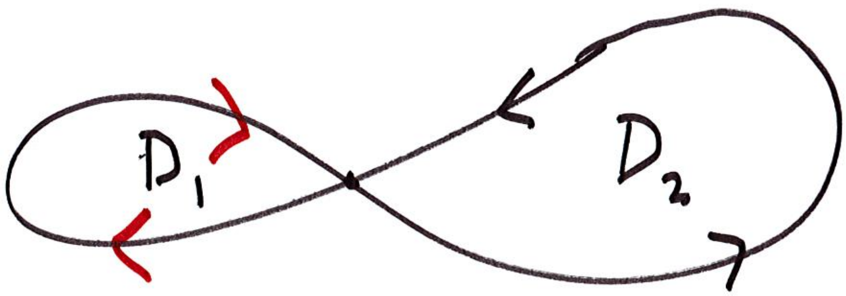


C = boundary of D

C is <u>positively</u> oriented

(Counterclockwise)

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$



<u>Recall that</u>

$$\int_C P\,dx + Q\,dy =$$

$$= \int \left(P(x(t), y(t))\, x'(t) + Q(x(t), y(t))\, y'(t) \right) dt$$

$$= \int_C \vec{F} \cdot d\vec{r} \qquad \vec{F} = P\vec{i} + Q\vec{j}.$$

If $\vec{F} = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$

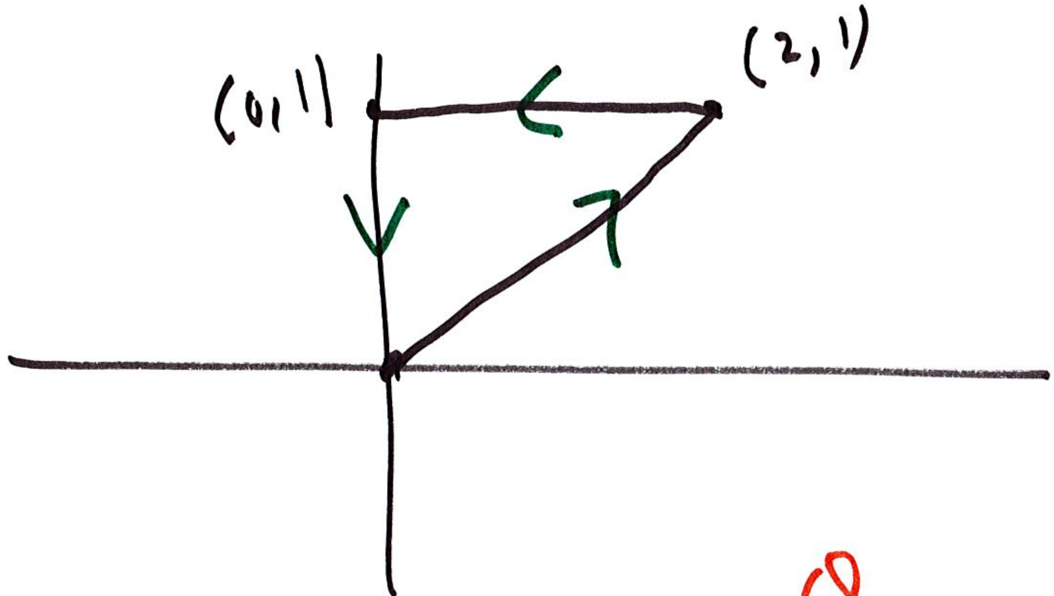
$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Examples:

1) Compute $\int_C (x^2+y^2)\,dx + (y^2-x^2)\,dy$

where C is the triangle with vertices $(0,0)$; $(2,1)$, $(0,1)$ positively oriented.

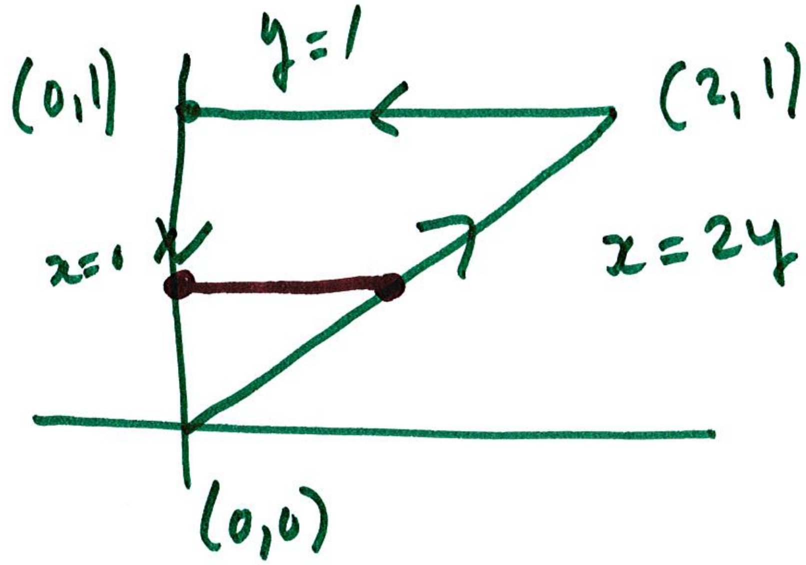


$$\oint_C \overbrace{(x^2+y^2)}^{P}\,dx + \overbrace{(y^2-x^2)}^{Q}\,dy$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D (-2x - 2y)\,dA.$$

$$= -2\iint_D (x+y)\,dA.$$



$$= -2\int_0^1\int_0^{2y}(x+y)\,dx\,dy$$

$$= -2\int_0^1\left[\frac{1}{2}(4y^2) + 2y^2\right]dy$$

$$= -2\int_0^1 4y^2\,dy = -8\int_0^1 y^2\,dy$$

$$= -8/3 .$$

One can compute areas using line integrals.

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Arrange P and Q such that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1.$$

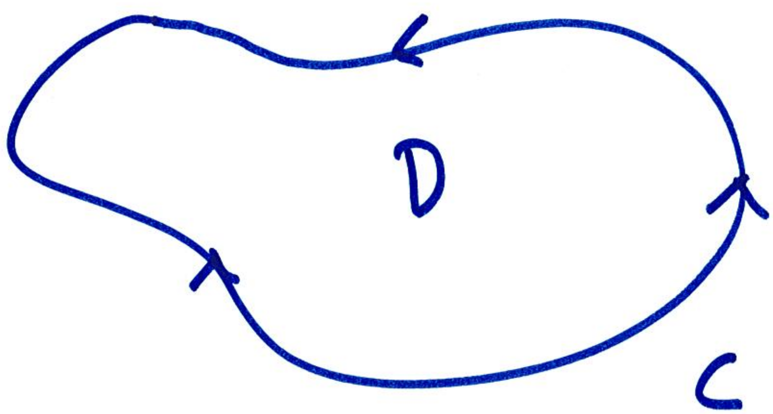
$$Q = \frac{1}{2}x\,; \quad P = -\frac{1}{2}y$$

$$\frac{\partial Q}{\partial x} = \frac{1}{2}\,; \quad \frac{\partial P}{\partial y} = -\frac{1}{2}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1.$$

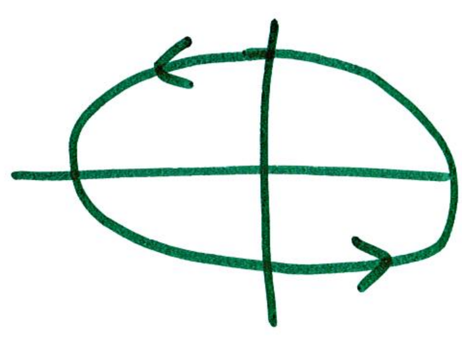
$$\int_C -\tfrac{1}{2} y\,dx + \tfrac{1}{2} x\,dy$$

$$= \iint_D dA = \text{area of } D.$$

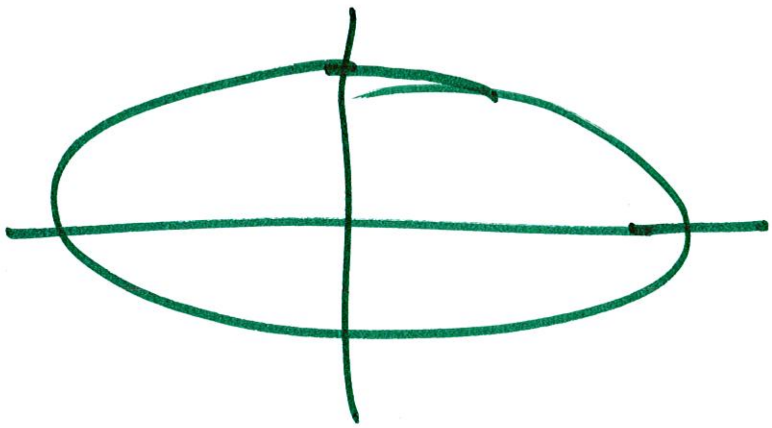


Compute the area of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Step 1: Parametrize the ellipse



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x = a\cos t, \quad y = b\sin t$$

$$0 \le t \le 2\pi$$

$$\int -\frac{1}{2} y\,dx + \frac{1}{2} x\,dy =$$

$$= \int_0^{2\pi} \left[-\frac{1}{2} b\sin t\,(-a\sin t) + \frac{1}{2} a\cos t\, b\cos t \right] dt$$

$$= \int_0^{2\pi} \left(\frac{1}{2} ab \cos^2 t + \frac{1}{2} ab \sin^2 t \right) dt$$

$$= \int_0^{2\pi} \frac{1}{2} ab \, dt = \frac{1}{2} ab \cdot 2\pi$$

$$= \pi ab.$$

In the case of a circle:

$$a = b = R$$

$$\underline{\text{Area} = \pi R^2}.$$