EXAM II. 2

April 11 8:00 P.M

Elliott

Lessons 18 - 29.

Same assigned seats

Lesson 31 Section 16.5

Curl and Divergence

(ROTATIONAL)

I. THE CURL OF A VECTOR FIELD.

$$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$$

P, Q, R are differentiable.

$$\text{Curl } \vec{F} = \vec{\nabla} \times \vec{F}$$

$$= \underbrace{\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)}\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\vec{j}$$

$$+ \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\vec{k}.$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \vec{j}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)$$

$$+ \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

Conservative Vector Fields

$$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$$

is conservative if

$$\vec{F} = \vec{\nabla} f.$$

$$\vec{\nabla} \times (\vec{\nabla} f) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right)$$

$$- \vec{j}\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right)$$

$$+ \vec{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right)$$

If f <u>has continuous</u> second order partial derivatives

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}, \text{ and so on}$$

When the second order partial derivatives are continuous

$$\vec{\nabla} \times (\vec{\nabla} f) = 0.$$

Conclusion: If

$$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$$

and $\vec{\nabla} \times \vec{F} \neq 0$, $\vec{F}$ is not conservative.

Provided P, Q, R have continuous partial derivatives.

If $\vec{F}$ were conservative

$\vec{F} = \vec{\nabla} f$ and $\vec{\nabla} \times \vec{F} = 0$

Examples: Find $\vec{\nabla} \times \vec{F}$

$$\vec{F} = e^{xy}\,\vec{i} + xy^2z\,\vec{j} + (z^3+y^2)\,\vec{k}.$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{xy} & xy^2z & (y^2+z^3) \end{vmatrix}$$

$$= \vec{i}\,(2y - xy^2) - \vec{j}\,(0 - 0)$$

$$+ \vec{k}\,(y^2z - xe^{xy})$$

$$\vec{F} = P\vec{i} + Q\vec{j} + 0\vec{k}.$$

$$P = P(x,y), \quad Q(x,y) = Q$$

$$\vec{\nabla} \times \vec{F} = 0\vec{i} + 0\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$\vec{\nabla} \times \vec{F} \cdot \vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

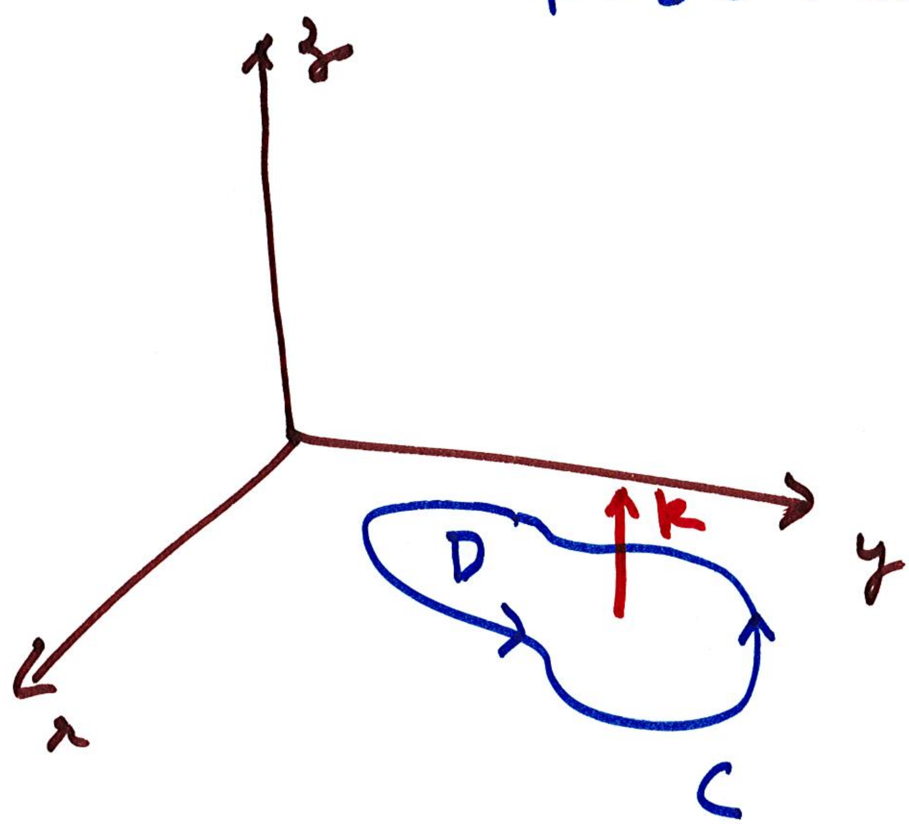
<u>Green's Theorem</u>

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \vec{\nabla} \times \vec{F} \cdot \vec{k} \, dA.$$

Green's theorem:

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

$$\vec{F} = P\vec{i} + Q\vec{j}$$



The DIVERGENT of a vector field.

$$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}.$$

$$\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot$$

$$\left(P\vec{i} + Q\vec{j} + R\vec{k}\right)$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

$$\boxed{\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}}$$

$$\boxed{\operatorname{Curl} \vec{F} = \vec{\nabla} \times \vec{F}}$$

Example: Find $\operatorname{div} \vec{F}$

$$\vec{F} = xyz^2 \vec{i} + 2e^x \cos y \, \vec{j} + z^3 xy \, \vec{k}.$$

$$\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} =$$

$$= \partial/\partial x \left(xyz^2\right) + \partial/\partial y \left(2e^x \cos y\right)$$

$$+ \partial/\partial z \left(z^3 xy\right) =$$

$$= yz^2 + (-1)\, 2e^x \sin y + 3z^2 xy$$

$$= yz^2 - 2e^x \sin y + 3z^2 xy.$$

The divergence of $\vec{F}$ is a function.

$$\text{div}(\vec{\nabla} f) = \vec{\nabla} \cdot (\nabla \vec{f})$$

$$= \text{div}\left(\frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} \right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f$$

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

$$\boxed{\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}}$$

Laplacian

<u>Example:</u> Wave equation:

$$\frac{\partial^2 f}{\partial t^2} - \Delta f = 0$$

<u>Heat Equation:</u>

$$\frac{\partial f}{\partial t} - \Delta f = 0$$