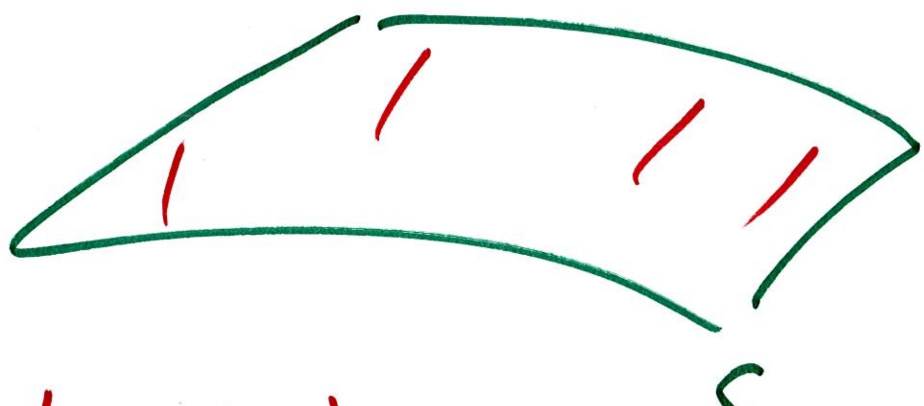
Exam 2.2 8:00 PM

Tonight in Elliott!!

Lesson 33 Section 16.6 Part II.

Parametric Surfaces and Areas

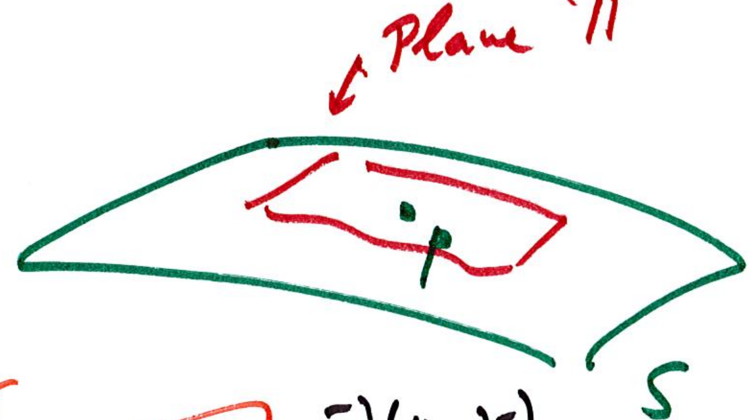


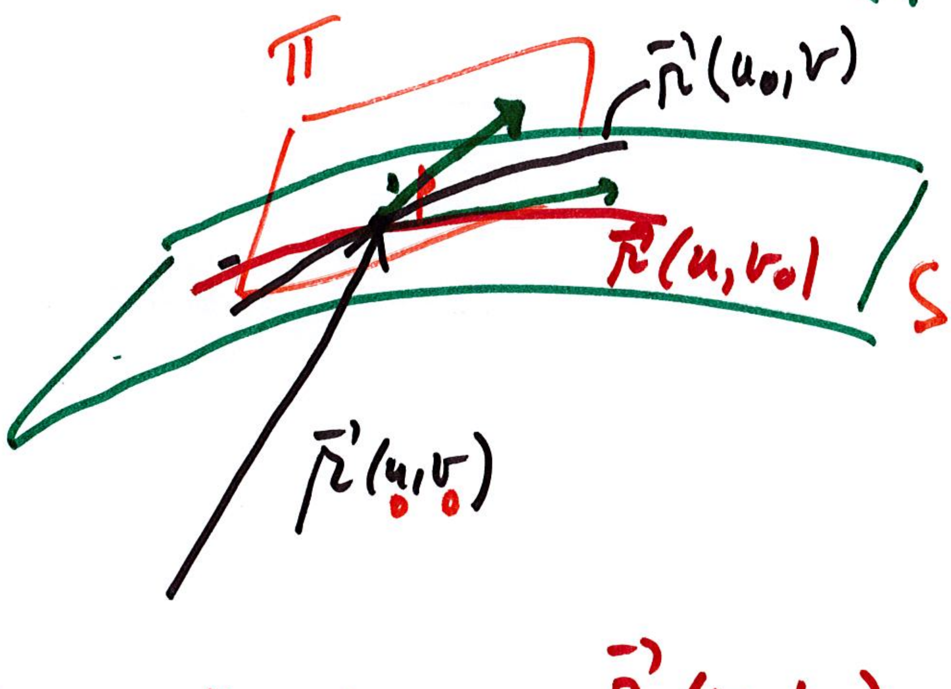
S is parametrized by

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}.$$

The plan is to use this to compute the area of S

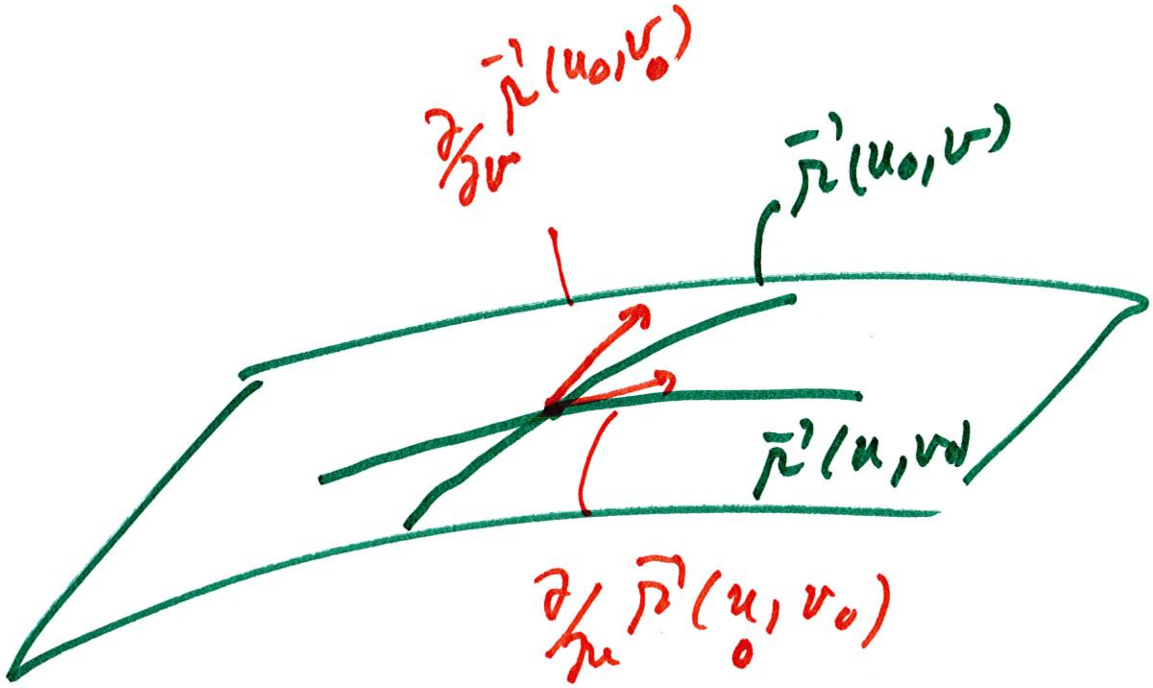
Step 1: Find an equation for the plane tangent to S at a point p in S





Fix $v = v_0$, vary u, $\vec{r}(u, v_0)$

$u = u_0$ vary v, $\vec{r}(u_0, v)$



The vectors $\dfrac{\partial \vec{r}}{\partial u}(u_0,v_0)$ and $\dfrac{\partial \vec{r}}{\partial v}(u_0,v_0)$ lie on the tangent plane.

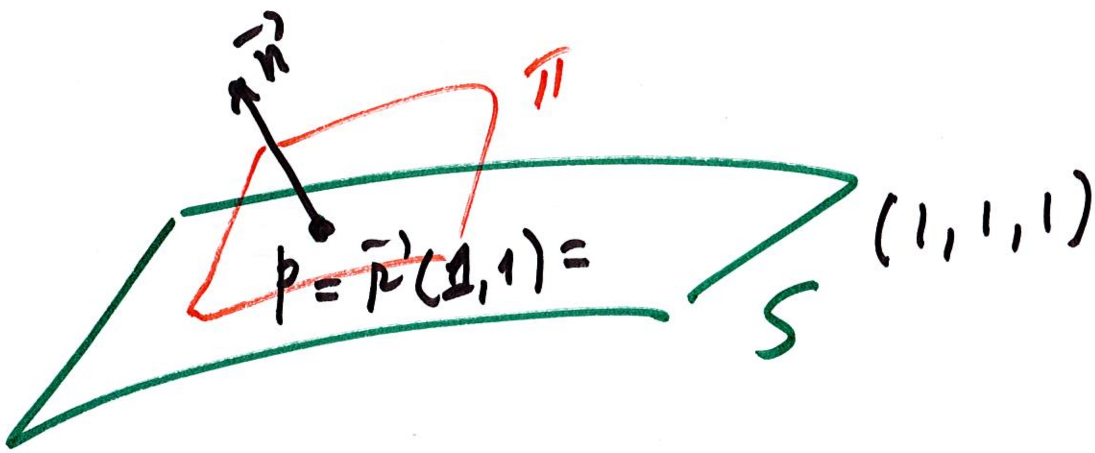
$\vec{n} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$

Once we find $\vec{n}$, we can find an equation for the plane.

Ex: Find an equation for the plane tangent to the surface parametrized by

$$\vec{r}(u,v) = u^2\vec{i} + u^2v^3\vec{j} + uv\vec{k}$$

when $u = v = 1$



$$\vec{n} = \frac{\partial \vec{r}}{\partial u}(1,1) \times \frac{\partial \vec{r}}{\partial v}(1,1)$$

$$\frac{\partial \vec{r}}{\partial u} = 2u\vec{i} + 2uv^3\vec{j} + v\vec{k}, \quad \frac{\partial \vec{r}}{\partial u}(1,1) = 2\vec{i} + 2\vec{j} + \vec{k}$$

$$\frac{\partial \vec{r}}{\partial v} = 3u^2v^2\vec{j} + u\vec{k} \qquad \frac{\partial \vec{r}}{\partial v}(1,1) = 3\vec{j} + \vec{k}.$$

$$\vec{n} = (2\vec{i} + 2\vec{j} + \vec{k}) \times (3\vec{j} + \vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 1 \\ 0 & 3 & 1 \end{vmatrix} = \vec{i}(-1) - \vec{j}(2) + \vec{k}(6)$$

$$= -\vec{i} - 2\vec{j} + 6\vec{k}.$$

<u>Equation of the plane</u>: The plane goes through $(1,1,1)$ with normal vector $-\vec{i} - 2\vec{j} + 6\vec{k}$.

$$\left((x-1)\vec{i} + (y-1)\vec{j} + (z-1)\vec{k} \right) \cdot (-\vec{i} - 2\vec{j} + 6\vec{k}) = 0$$

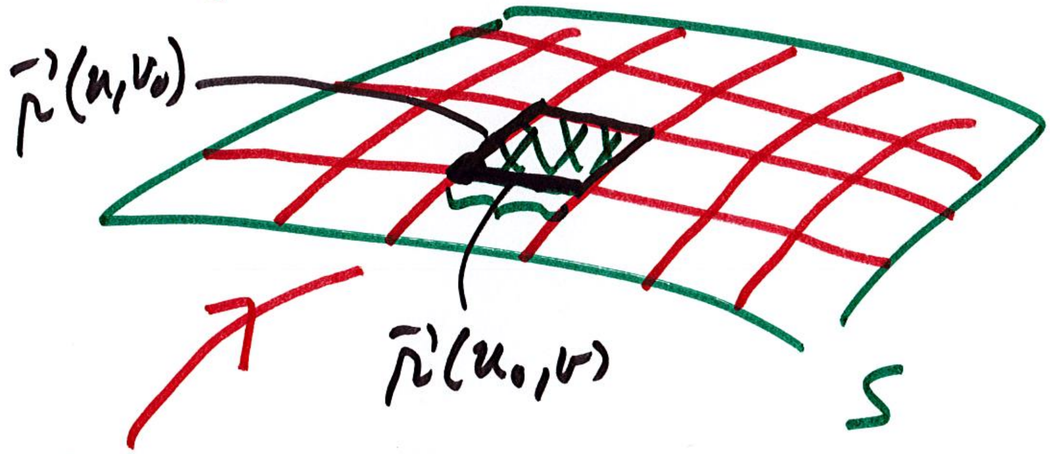
$$\boxed{-(x-1) - 2(y-1) + 6(z-1) = 0}$$

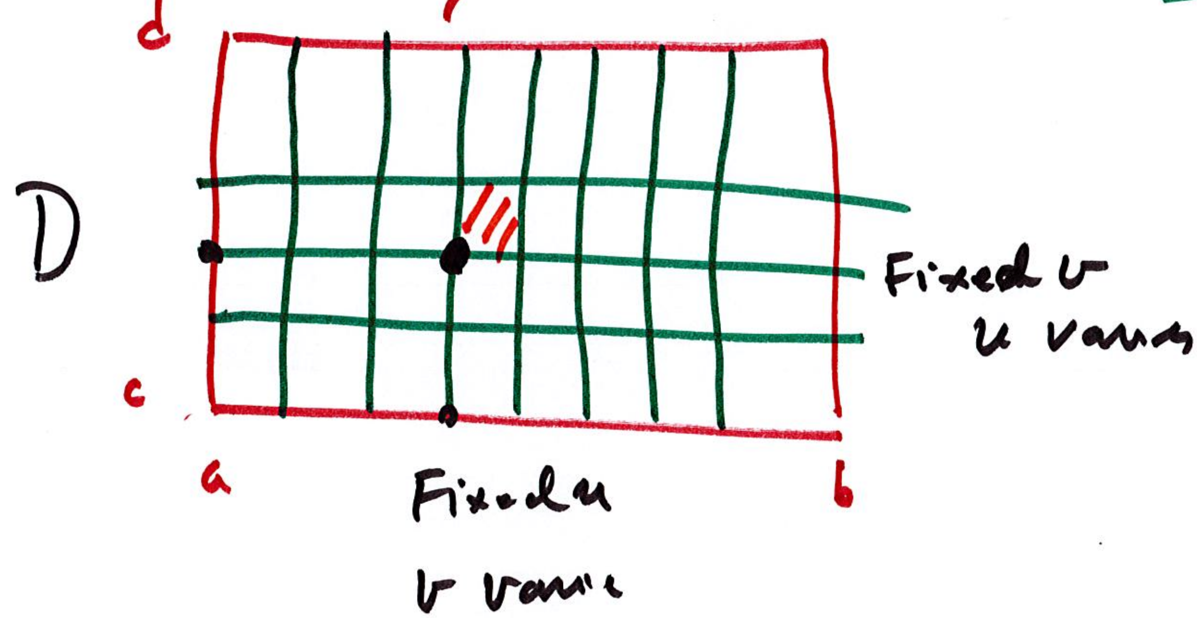
Compute the area of S

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

$a \le u \le b$

$c \le v \le d.$





Area of the parallelogram on the surface

$$\left| \frac{\partial \vec{r}}{\partial u}(u_0,v_0)\,du \times \frac{\partial \vec{r}}{\partial v}(u_0,v_0)\,dv \right|$$

$$= \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du\,dv.$$

Area of the Surface: S

$$A(S) = \iint_D \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du\, dv$$

D is the rectangle: $a \le u \le b$

$c \le v \le d$.

Examples: $x = u^2,\ y = uv,\ z = \frac{1}{2} v^2$

$0 \le u \le 2$

$0 \le v \le 1$.

Solution: $\vec{r}(u,v) = u^2 \vec{i} + uv \vec{j} + \frac{1}{2} v^2 \vec{k}$.

$$\frac{\partial \vec{r}}{\partial u} = 2u \vec{i} + v \vec{j}, \quad \frac{\partial \vec{r}}{\partial v} = u \vec{j} + v \vec{k}.$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & v & 0 \\ 0 & u & v \end{vmatrix}$$

$$= \vec{i}\,(v^2) - \vec{j}\,(2uv) + \vec{k}\,(2u^2)$$

$$\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|^2 = v^4 + 4u^2v^2 + 4u^4$$

$$= (v^2 + 2u^2)^2$$

$$\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| = v^2 + 2u^2.$$

$0 \le u \le 2$

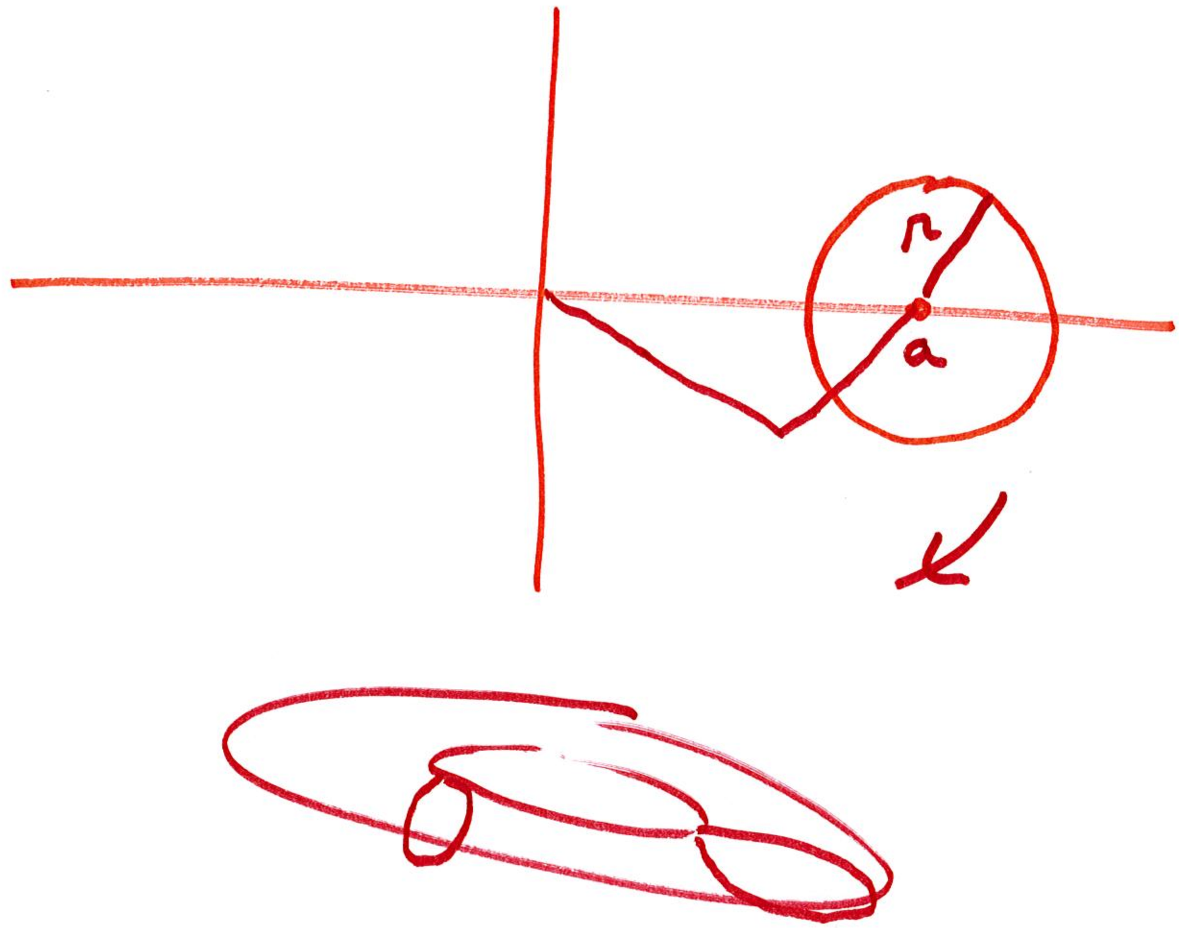
$0 \le v \le 1$

$$\text{Area of } S = \int_0^1 \int_0^2 (v^2 + 2u^2)\, du\, dv.$$

... Finish this. . . .

Surface area of a doughnut

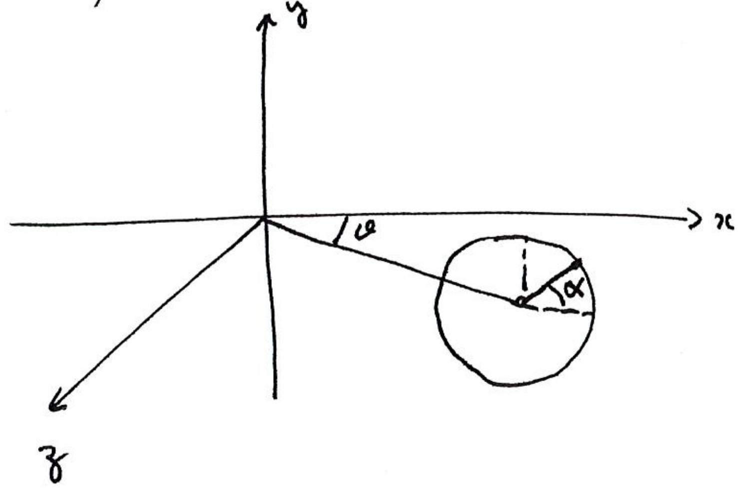
(Torus)



$$(2\pi r)(2\pi a) = \underline{(4\pi)^2 a r}$$

$$S = \int_0^{\pi}\int_0^{1} \sqrt{1+u^2}\, du\, dv = \pi \int_0^{1} \sqrt{1+u^2}\, du$$

Torus (Doughnut)



$$x = (a + r\cos\alpha)\cos\theta, \quad y = r\sin\alpha$$

$$z = (a + r\cos\alpha)\sin\theta. \qquad 0 \le \alpha \le 2\pi$$

$$0 \le \theta \le 2\pi.$$

$$\frac{\partial \vec{r}}{\partial \theta} = -(a + r\cos\alpha)\sin\theta\, \vec{i} + (a + r\cos\alpha)\cos\theta\, \vec{k}$$

$$\frac{\partial \vec{r}}{\partial \alpha} = (-r\sin\alpha)\cos\theta\, \vec{i} + r\cos\alpha\, \vec{j}$$

$$- r\sin\alpha\sin\theta\, \vec{k}$$

$$\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \alpha} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -(a+r\cos\alpha)\sin\theta & 0 & (a+r\cos\alpha)\cos\theta \\ -r\sin\alpha\cos\theta & r\cos\alpha & -r\sin\alpha\sin\theta \end{vmatrix}$$

$$= r\cos\alpha\,(a+r\cos\alpha)\cos\theta\,\vec{i}$$

$$- \vec{j}\left(r\sin\alpha\,(a+r\cos\alpha)\sin^2\theta + r\sin\alpha\,(a+r\cos\alpha)\cos^2\theta\right)$$

$$- \vec{k}\; r\cos\alpha\,(a+r\cos\alpha)\sin\theta .$$

$$= r\cos\alpha\,(a+r\cos\alpha)\left(\cos\theta\,\vec{i} + \sin\theta\,\vec{k}\right)$$

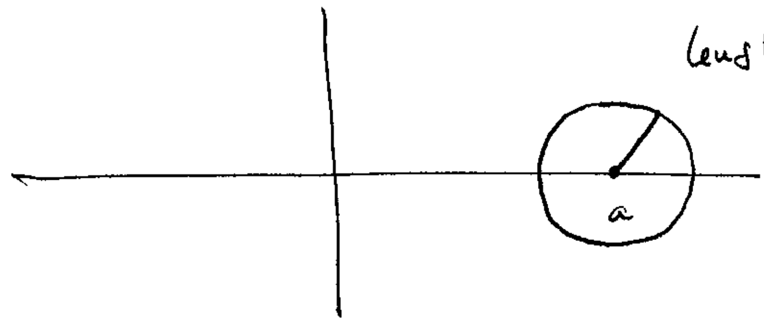
$$- r\sin\alpha\,(a+r\cos\alpha)\,\vec{j}$$

$$\left|\frac{\partial r}{\partial \theta} \times \frac{\partial r}{\partial \alpha}\right|^2 = r^2\,(a+r\cos\alpha)^2$$

$$\boxed{\left|\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial r}{\partial \alpha}\right| = r\,(a+r\cos\alpha)}$$

$$S = \int_0^{2\pi} \int_0^{2\pi} r(a + r\cos\alpha)\, d\alpha\, d\omega$$

$$= \underline{(2\pi)^2 r a}.$$



length of circle

$2\pi r$