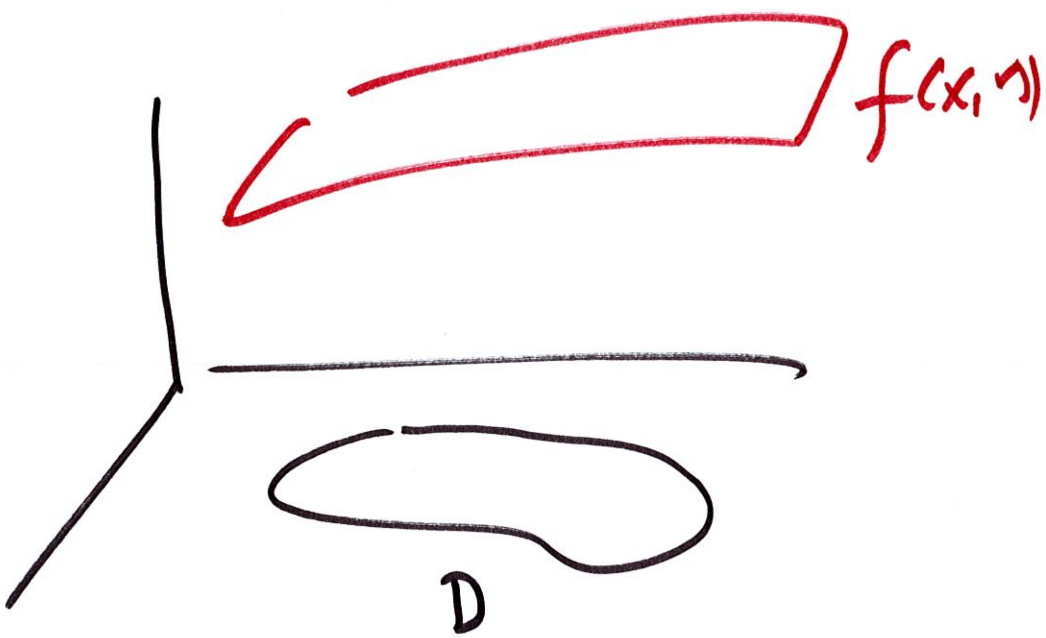
Lesson 34 Section 16.7 Part I

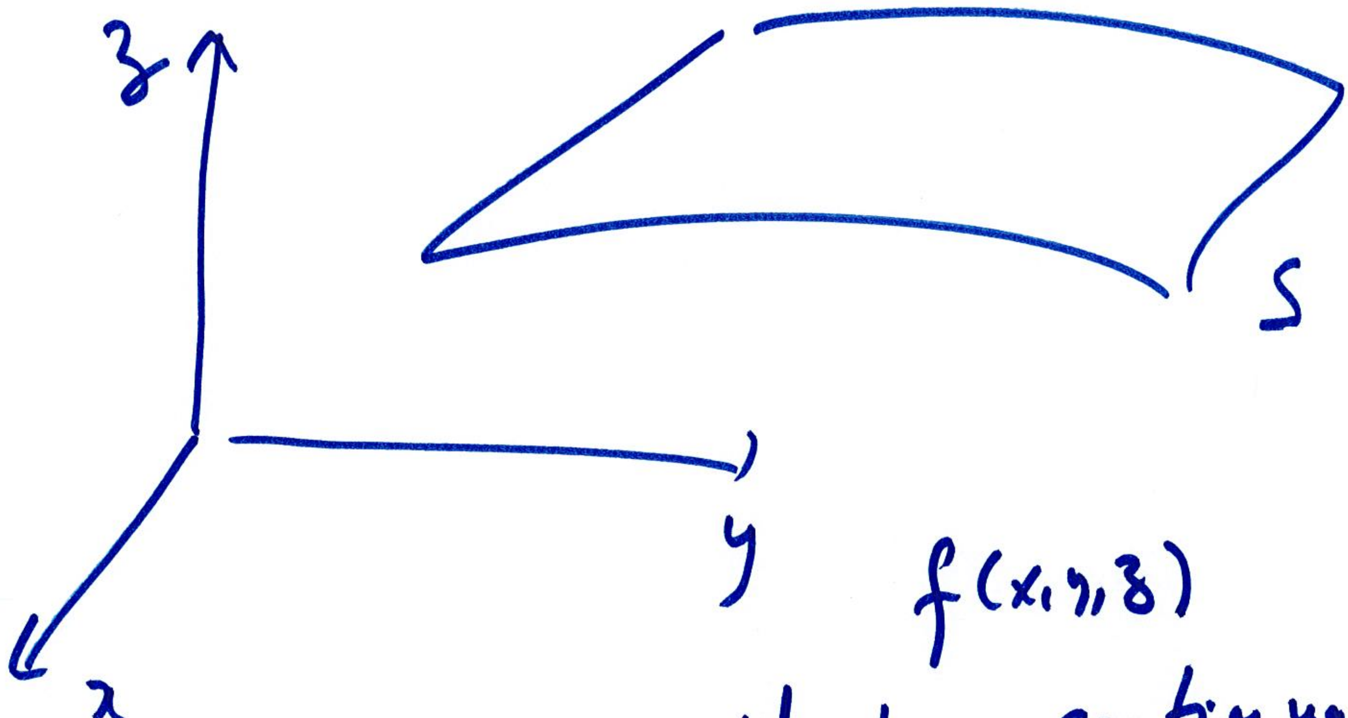
Surface Integrals

We studied double integrals



$$\iint_D f(x,y)\, dA .$$

Surface integrals is a generalization of double integrals



$f(x,y,z)$

which is continuous

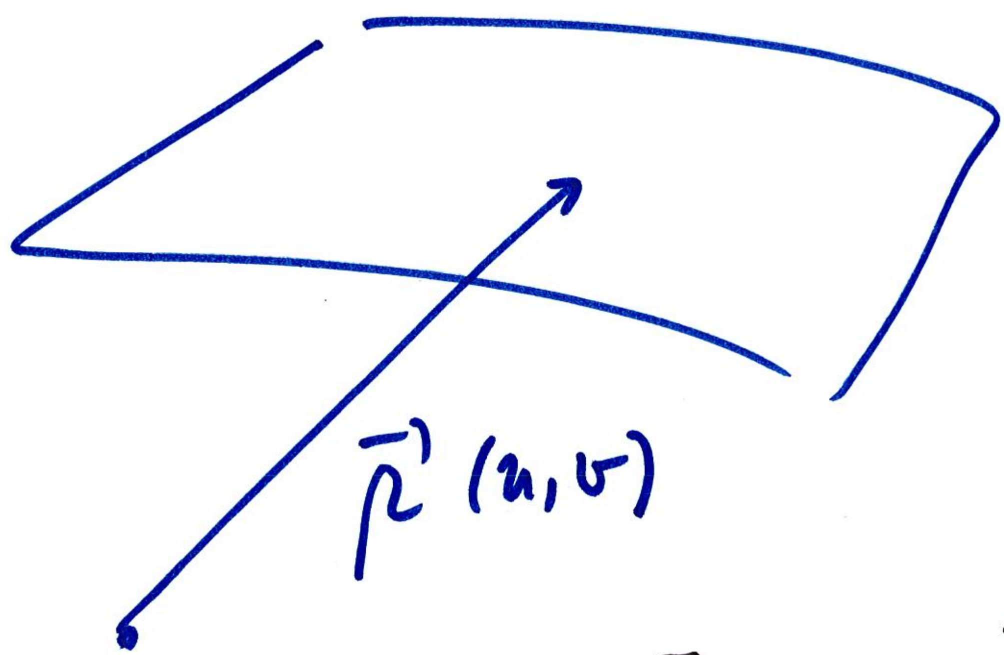
We want to define

$$\iint_S f(x,y,z)\, dS \,.$$

Surface area $f = 1$.

$$\iint_S 1 \, dS = \text{Area of the Surface.}$$

We defined this in the following way:



1) $\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$.

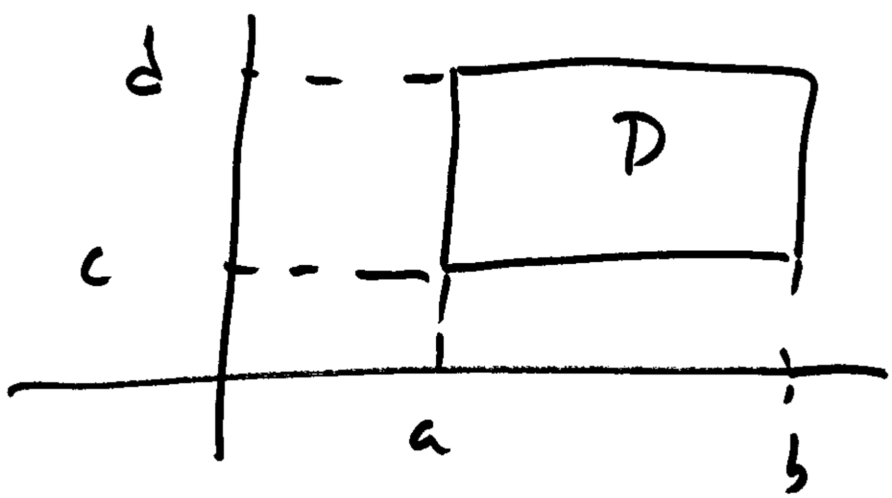
parametrizes S. $a \le u \le b$

$c \le v \le d$.

2) Compute:

$$\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|.$$

3) $$\text{Area} = \iint_D \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dA$$



Based on the same construction,

$$\iint_S f \, dS =$$

$$= \iint_D \underbrace{f\left(x(u,v), y(u,v), z(u,v)\right)} \underbrace{\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|} \, dA$$

<u>Examples</u> :

(1) Compute $\displaystyle\iint_S xyz \, dS$

S is the surface with parametric equations

$x = u \cos v, \quad y = u \sin v$

$z = u \qquad 0 \le u \le 1$

$\qquad\qquad 0 \le v \le \pi/2.$

Step 1: Compute $\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|$.

$$\vec{r}(u,v) = u\cos v\,\vec{i} + u\sin v\,\vec{j} + u\,\vec{k}.$$

$$\frac{\partial \vec{r}}{\partial u} = \cos v\,\vec{i} + \sin v\,\vec{j} + \vec{k}$$

$$\frac{\partial \vec{r}}{\partial v} = -u\sin v\,\vec{i} + u\cos v\,\vec{j}$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix}$$

$$= \vec{i}\,(-u\cos v) - \vec{j}\,(u\sin v)$$
$$+ \vec{k}\,(u\cos^2 v + u\sin^2 v)$$
$$= -u\cos v\,\vec{i} - u\sin v\,\vec{j} + u\,\vec{k}.$$

$$\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial r}{\partial v} \right|^2 = u^2 \cos^2 v + u^2 \sin^2 v + u^2$$

$$= 2u^2$$

$$\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| = u\sqrt{2}$$

2) $f(x, y, z) = xyz$

$$f(x(u,v), y(u,v), z(u,v))$$

$$= (u\cos v).(u\sin v).u$$

$$= u^3 \sin v \cos v.$$

3) $\displaystyle\iint_S \cancel{f}\, xyz\, dS = \sqrt{2}\int_0^{\pi/2}\int_0^1 u^3 \sin v \cos v . u \, du\, dv$

$$= \sqrt{2} \int_0^{\pi/2} \int_0^1 u^4 \sin v \cos v \, du \, dv.$$

$$= \sqrt{2} \cdot \frac{1}{5} \int_0^{\pi/2} \sin v \cos v \, dv$$

$\sin v = t$

$dt = \cos v \, dv$

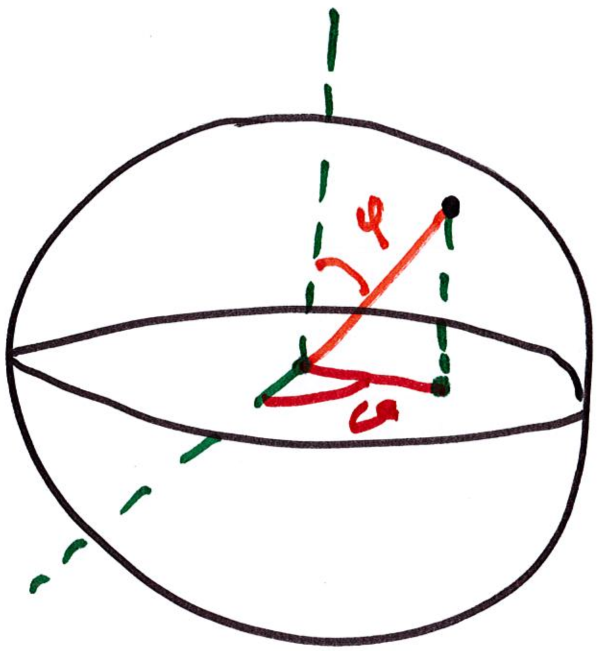
$$\int_0^{\pi/2} \sin v \cos v \, dv = \int_0^1 t \, dt$$

$$= 1/2$$

<u>Conclusion:</u> $\iint_S xyz \, dS = \frac{\sqrt{2}}{10}.$

$$\iint_S x^2\, dS \qquad S \text{ is the sphere } x^2+y^2+z^2=1.$$

Step 1: Parametrize S.



$x = \rho \sin\varphi \cos\theta$, but $\rho = 1$

$\boxed{x = \sin\varphi\cos\theta.}$, $y = \sin\varphi\sin\theta$

$z = \cos\varphi.$

$$\vec{r}'(\theta, \varphi) = \sin\varphi\cos\theta\,\vec{i}' + \sin\varphi\sin\theta\,\vec{j}' + \cos\varphi\,\vec{k}'.$$

<u>Step 2</u>: Compute $\left|\frac{\partial \vec{r}'}{\partial \theta} \times \frac{\partial \vec{r}'}{\partial \varphi}\right|$.

$$\frac{\partial \vec{r}'}{\partial \theta} = -\sin\varphi\sin\theta\,\vec{i}' + \sin\varphi\cos\theta\,\vec{j}'$$

$$\frac{\partial \vec{r}'}{\partial \varphi} = \cos\varphi\cos\theta\,\vec{i}' + \cos\varphi\sin\theta\,\vec{j}' - \sin\varphi\,\vec{k}'.$$

$$\frac{\partial \vec{r}'}{\partial \theta} \times \frac{\partial \vec{r}'}{\partial \varphi} = \begin{vmatrix} \vec{i}' & \vec{j}' & \vec{k}' \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \end{vmatrix}$$

$$= \vec{i}'(-\sin^2\varphi\cos\theta) - \vec{j}'(\sin^2\varphi\sin\theta) + \vec{k}'(-\underbrace{\sin\varphi\cos\varphi}\sin^2\theta - \underbrace{\sin\varphi\cos\theta}\cos^2\theta)$$

$$= -\sin^2\varphi\cos\theta\,\vec{i} - \sin^2\varphi\sin\theta\,\vec{j} - \sin\varphi\cos\varphi\,\vec{k}.$$

$$\left|\frac{\partial\vec{r}}{\partial\varphi}\times\frac{\partial\vec{r}}{\partial\theta}\right|^2 = \sin^4\varphi\,\underbrace{\cos^2\theta} + \sin^4\varphi\,\underbrace{\sin^2\theta} + \sin^2\varphi\cos^2\varphi$$

$$= \sin^4\varphi + \sin^2\varphi\cos^2\varphi$$

$$= \sin^2\varphi\,(\underbrace{\sin^2\varphi + \cos^2\varphi}_{=1})$$

$$= \sin^2\varphi.$$

$$\boxed{\left|\frac{\partial\vec{r}}{\partial\varphi}\times\frac{\partial\vec{r}}{\partial\theta}\right| = \sin\varphi}$$

$$\iint x^2 \, dS =$$

$$= \int_0^{2\pi}\int_0^{\pi} (\sin\varphi \cos\theta)^2 \cdot \overbrace{\left| \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \varphi} \right|}^{\sin\varphi} d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} (\sin\varphi \cos\theta)^2 \sin\varphi \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^3\varphi \cdot \underline{\cos^2\theta} \, d\varphi \, d\theta$$

$$= \frac{4\pi}{3}.$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\int_0^{2\pi} \cos^2\theta \, d\theta = \int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2} \right) d\theta$$

$$= \pi.$$

$$\int_0^{\pi} \sin^3 \varphi \, d\varphi =$$

$$= \int_0^{\pi} \sin\varphi \cdot \sin^2\varphi \, d\varphi$$

$$= \int_0^{\pi} \underline{\sin\varphi} \, (1 - \cos^2\varphi) \, \underline{d\varphi}$$

$\cos\varphi = u \qquad du = -\sin\varphi \, d\varphi$

$$= \int_1^{-1} (1 - u^2)(-du)$$

$$= \int_{-1}^{1} (1-u^2) \, du = u - \frac{u^3}{3} \Big|_{-1}^{1}$$

$$= \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = 2\left(1 - \frac{1}{3}\right)$$

$$= 4/3 .$$

$$\boxed{\left| \frac{\partial \bar{r}'}{\partial \varphi} \times \frac{\partial \bar{r}'}{\partial \theta} \right| = \sin \varphi}$$