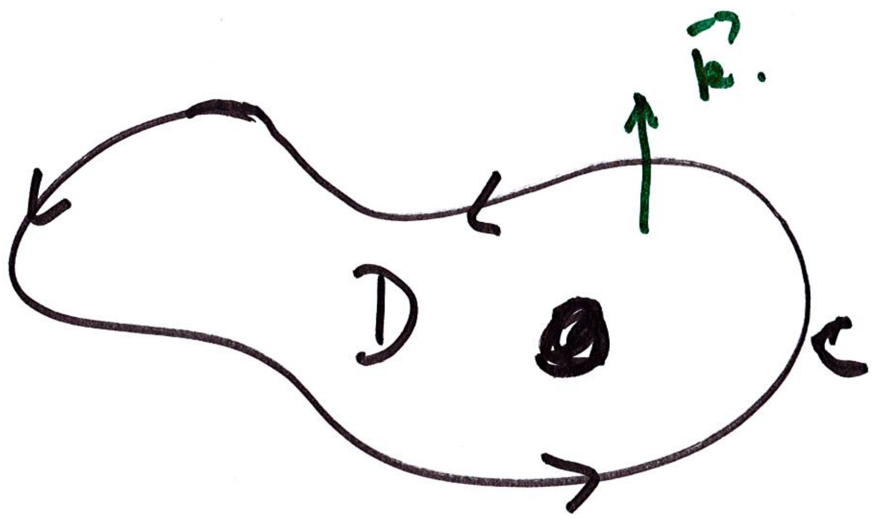
Lesson 36 Section 16.8

Stokes' Theorem

Is a generalization of Green's theorem.

Green's theorem



C = Closed curve, no self intersections.

C encloses D.

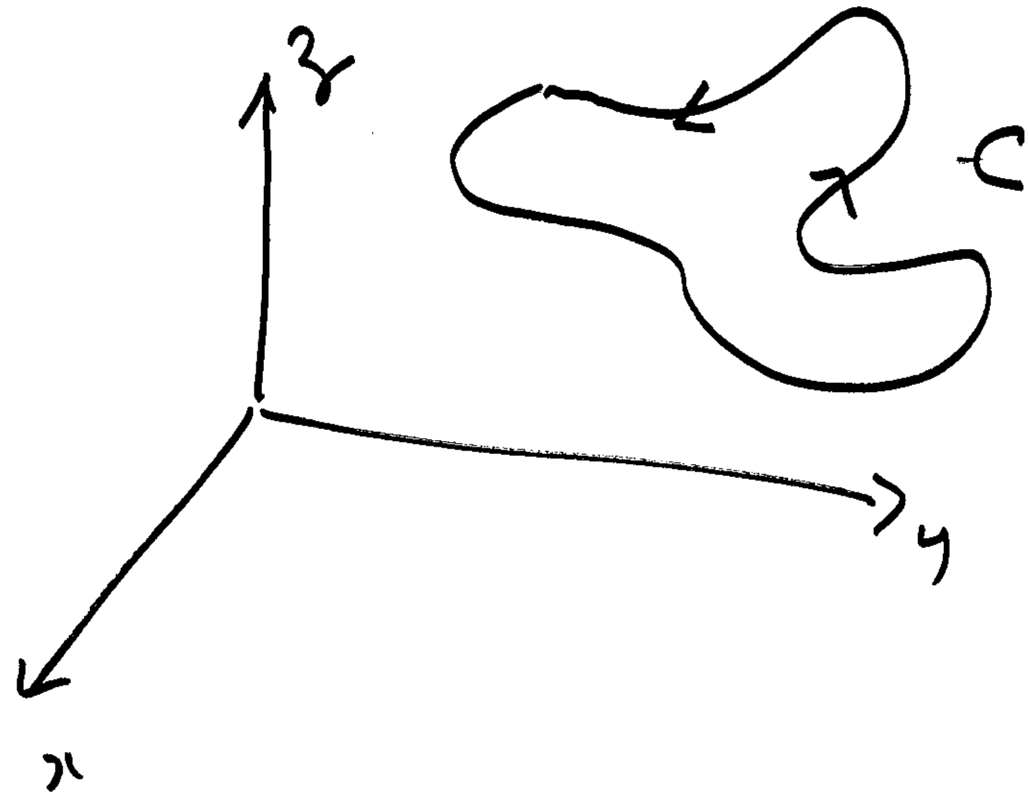
If C is positively oriented (Counterclockwise)

$$\int_C \vec{F} \cdot dr = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\text{Curl } \vec{F}} dA$$

$$\vec{F} = P\vec{i} + Q\vec{j} \qquad = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}.$$

Partial derivatives of P and Q are continuous.

<u>Question</u>: What is the analogue of this in 3-dimensions?



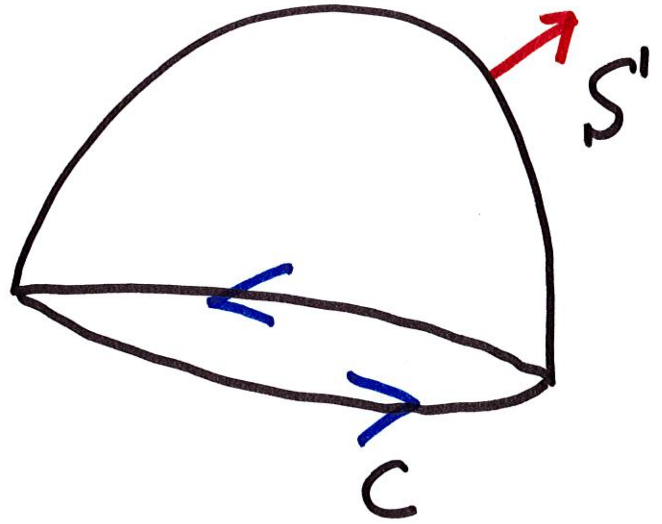
C is a closed curve in 3-dimensions.

no self intersections

$$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}.$$

$$\int_C \vec{F} \cdot d\vec{r} \overset{?}{=} \iint$$

surface integral



(1) S is a smooth surface and C is the boundary of S

(2) S is oriented and the orientation of S induces an orientation of C according to the right hand rule.

(3) $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

P, Q, R have continuous partial derivatives

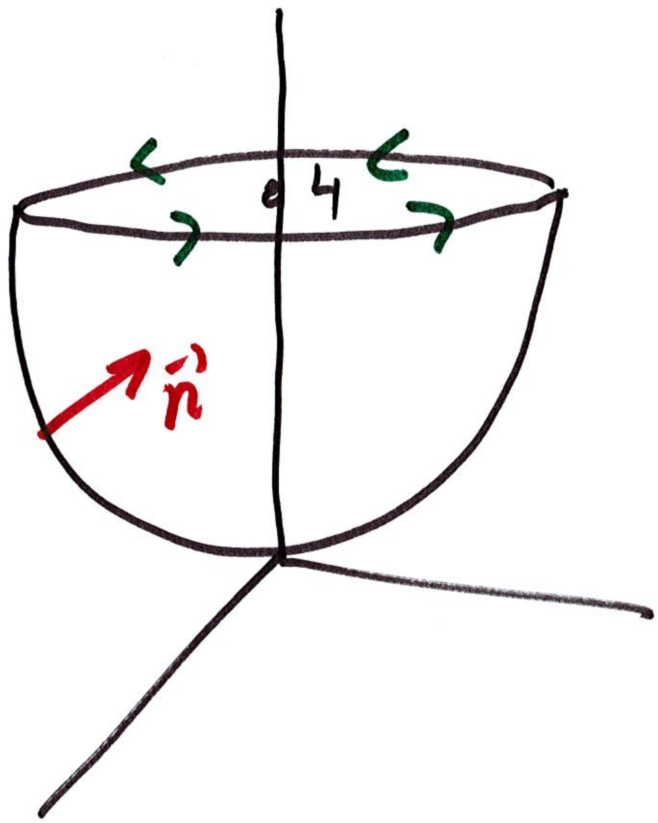
$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl}\, \vec{F}) \cdot \vec{n}\, dS$$

Examples:

(1) $\vec{F} = y\vec{i} + x\vec{j} + z^2\vec{k}$

S

Oriented upward



$z = x^2 + y^2$ below $z = 4$

Compute $\displaystyle\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS$

$\vec{\nabla} \times \vec{F} = \text{Curl}\,\vec{F}.$

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS = \int_C \vec{F} \cdot d\vec{r}.$$

C is the curve

$$x^2 + y^2 = 4, \quad z = 4.$$

$$x = 2\cos t, \quad y = 2\sin t, \quad z = 4$$

$$\vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + 4\vec{k}. \quad 0 \le t \le 2\pi$$

$$\int \vec{F} \cdot d\vec{r} = \int_0^{2\pi} F(r(t)) \cdot \vec{r}'(t)\,dt$$

$$\vec{F}(r(t)) = 2\sin t\,\vec{i} + 2\cos t\,\vec{j} + 16\vec{k}.$$

$$\vec{r}'(t) = -2\sin t\,\vec{i} + 2\cos t\,\vec{j}$$

$$\vec{F} \cdot \vec{r}' = 4\cos^2 t - 4\sin^2 t$$

$$= 4(\cos^2 t - \sin^2 t) = 4\cos 2t.$$

$$\int \vec{F} \cdot d\vec{r} = \int_0^{2\pi} 4 \cos 2t \, dt = 0.$$

Compute $\iint \vec{\nabla} \times \vec{F} \cdot \vec{n}\, dS$

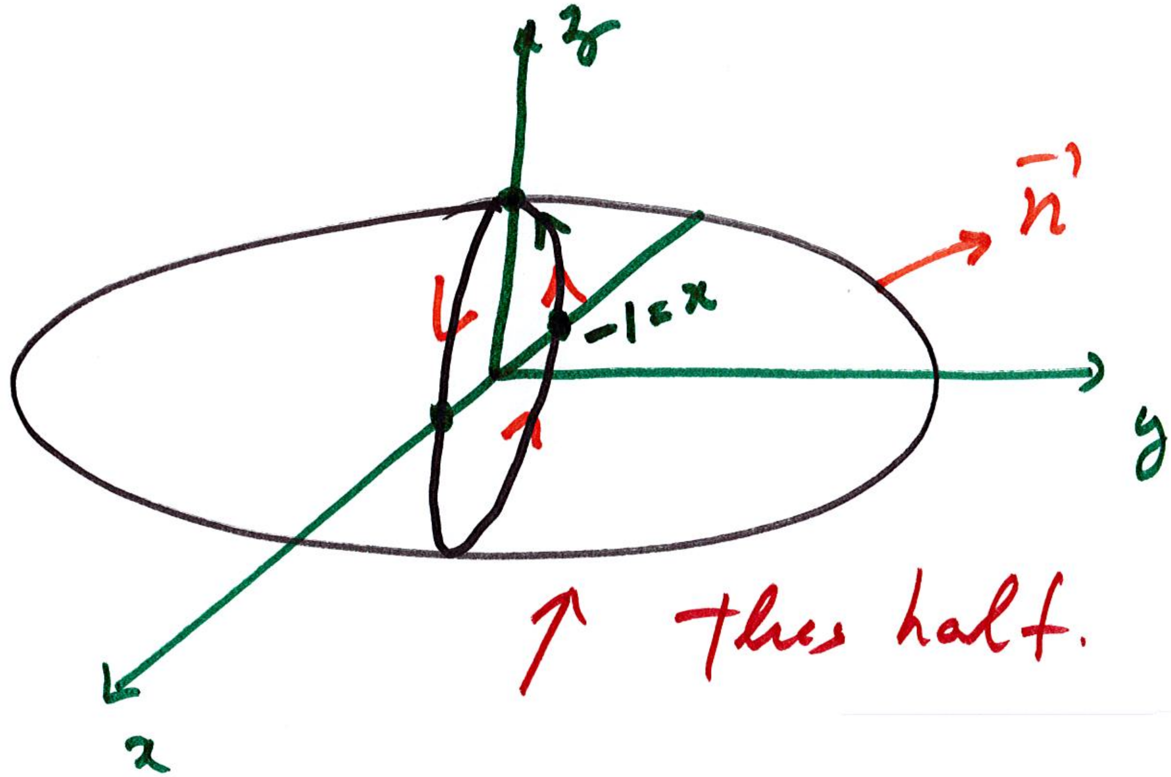
S = half of the ellipsoid

$$4x^2 + y^2 + 4z^2 = 4$$

to the right of the xz-plane

oriented in the direction of the positive y-axis.

$$\vec{F} = e^{xy}\vec{i} + e^{xz}\vec{j} + x\vec{k}.$$



$$\iint_S \vec{\nabla}\times\vec{F}\cdot\vec{n}\,dS = \underbrace{\int_C \vec{F}\cdot dr}_{\text{Compute this instead}}.$$

<u>Curve:</u> $4x^2+y^2+4z^2=4$

$$y=0$$

$$4x^2+4z^2=4\ ,\quad y=0$$

$$\boxed{x^2+z^2=1\ ,\quad y=0}$$

$x = -\cos t \qquad z = \sin t$

$y = 0$

$$\int \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$\vec{r}(t) = -\cos t\, \vec{i} + \sin t\, \vec{k}$$

$$F(\vec{r}(t)) = \vec{i} + e^{-\cos t \sin t}\, \vec{j} - \cos t\, \vec{k}.$$

$$\vec{r}\,'(t) = \sin t\, \vec{i} + \cos t\, \vec{k}.$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \sin t - \cos^2 t.$$

So

$$\int \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (\sin t - \cos^2 t)\, dt$$

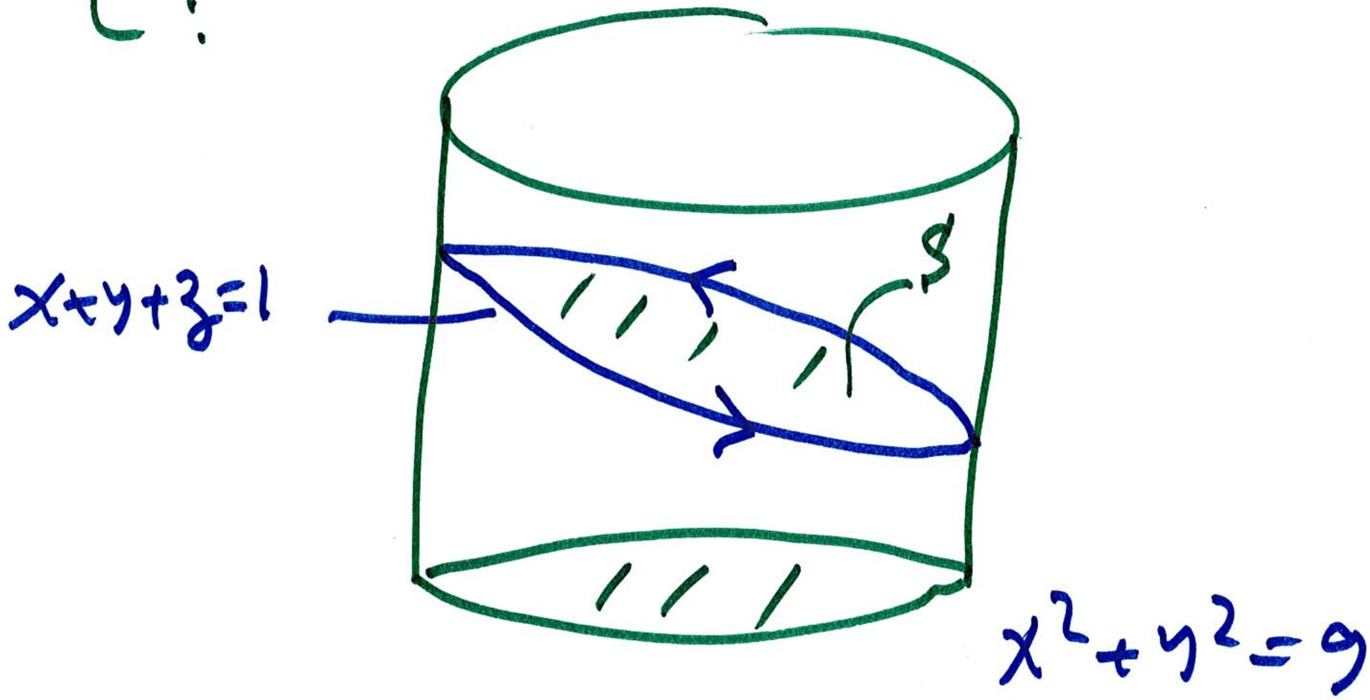
$$= -\int_0^{2\pi} \cos^2 t\, dt.$$

$$= -\int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt$$

$$= -\frac{1}{2}\int_0^{2\pi} \left(1 + \underbrace{\cos 2t}\right) dt$$

$$= -\pi.$$

C :



$$\int_C \vec{F} \cdot d\vec{r} \quad \text{where}$$

$$\vec{F} = x^2 z\,\vec{i} + x y^2 \vec{j} + z^2 \vec{k}.$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \vec{\nabla} \times \vec{F} \cdot \vec{n}\, dS$$

$S =$ Surface of the plane $x + y + z = 1$ above the circle

$$x^2 + y^2 = 9$$

Parametrize S:

$$\vec{r}(u,v) = u\vec{i} + v\vec{j} + (1-u-v)\vec{k}$$

Compute

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \vec{i} + \vec{j} + \vec{k}$$

$$\iint_{x^2+y^2 \le 9} (\nabla \times \vec{F}) \cdot (\vec{i} + \vec{j} + \vec{k}) \, dA.$$