MA 261 FINAL EXAM

MONDAY APRIL 30

1:00 P.M TO 3:00 PM

ELLIOTT HALL.

20 PROBLEMS = 200 POINTS

---

14) $f(x,y) = \cos(xy)$

$$\frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial f}{\partial y} = -\sin(xy) \cdot x = -x\sin(xy)$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\sin(xy) - x \cdot \cos(xy)\, y$$
$$= -\sin(xy) - xy\cos(xy).$$

Compute div(grad f).

$$\operatorname{grad} f = -y \sin(xy)\,\vec{i} - x \sin(xy)\,\vec{j}.$$

$$\operatorname{div}(\operatorname{grad} f) = \frac{\partial}{\partial x}\left(-y \sin(xy)\right) + \frac{\partial}{\partial y}\left(-x \sin(xy)\right)$$

$$F = P\vec{i} + Q\vec{j} + R\vec{k}$$

$$\operatorname{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$= -y^2 \cos(xy) - x^2 \cos xy$$

$$= -(x^2 + y^2)\cos(xy).$$

15) $\left[xy^2 + 3z = \cos(z^2) \right.$

Find $\dfrac{\partial z}{\partial x}$.

Solution 1 (Trust your memory)

$$F(x, y, z) = 0$$

$$\frac{\partial z}{\partial x} = - \frac{\partial F / \partial x}{\partial F / \partial z}.$$

$$F(x, y, z) = x y^2 + 3z - \cos(z^2) = 0$$

$$\boxed{\frac{\partial z}{\partial x} = - \frac{y^2}{3 + 2z \sin(z^2)}}$$

$$\frac{\partial z}{\partial x} = - \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

$$\boxed{\frac{\partial z}{\partial x} = - \frac{2xy}{3 + 2z \sin(z^2)}}$$

Solution 2: Use the ~~the~~ Chain rule and find the derivative.

$$x y^2 + 3z = \cos(z^2) \qquad z = z(x,y)$$

$$x y^2 + 3 z(x,y) = \cos\left(z^2(x,y)\right)$$

Take the derivative of this equation in x:

$$y^2 + 3\frac{\partial z}{\partial x} = -\sin(z^2) \cdot 2z\frac{\partial z}{\partial x}$$

now solve for $\frac{\partial z}{\partial x}$

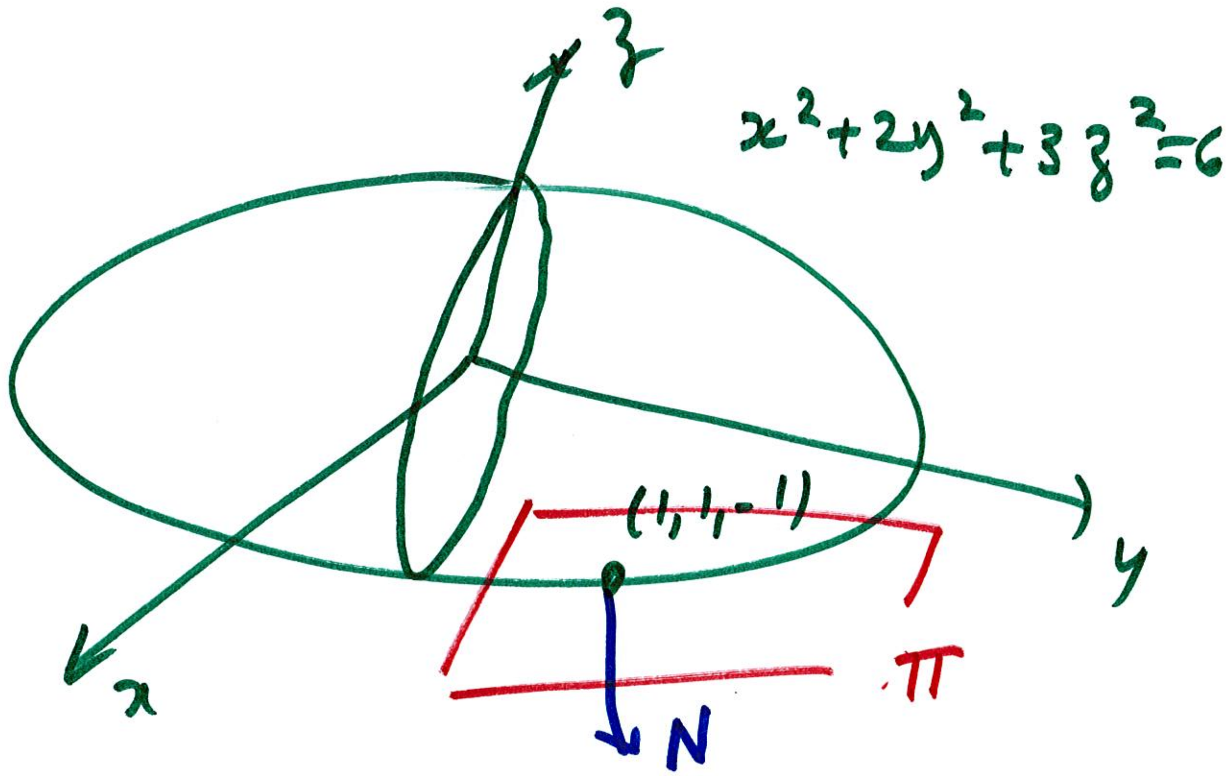
$$y^2 = -\sin(z^2) \cdot 2z\frac{\partial z}{\partial x} - 3\frac{\partial z}{\partial x}$$

$$y^2 = -\frac{\partial z}{\partial x}\left(3 + 2z\sin(z^2)\right)$$

$$\boxed{\frac{\partial z}{\partial x} = -\frac{y^2}{3 + 2z\sin(z^2)}}$$

#21) $x^2 + 2y^2 + 3z^2 = 6$

at $(1, 1, -1)$



Find a vector normal to π

Take $N = \nabla f(1, 1, -1)$

$f = x^2 + 2y^2 + 3z^2$.

$$\nabla f = 2x\vec{i} + 4y\vec{j} + 6z\vec{k}$$

$$N = \nabla f(1,1,-1) = 2\vec{i} + 4\vec{j} - 6\vec{k}$$

The plane that goes through $(1,1,-1)$ and is normal to N is given by

$$\left((x-1)\vec{i} + (y-1)\vec{j} + (z+1)\vec{k} \right) \cdot N = 0$$

$$\boxed{2(x-1) + 4(y-1) - 6(z+1) = 0}$$

$$2x + 4y - 6z - 12 = 0$$

$$2x + 4y - 6z = 12$$

$$\boxed{x + 2y - 3z = 6}$$

Variations of this problem.

(1) The intersection of the tangent plane with the plane $y=0$ is

$$\begin{cases} x+2y-3z=6 \\ y=0 \end{cases}$$

$$\begin{cases} x-3z=6 \\ y=0 \end{cases}$$

(2) If the point $(0,1,z)$ is on the plane, find z.

$x+2y-3z=6, \quad 2-3z=6, \; -3z=4$

$z=-4/3.$

24) $f(x,y) = 2x^3 - 6xy - 3y^2$

Step 1: Find the critical points.

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial x} = 6x^2 - 6y = 0 \; ; \; \frac{\partial^2 f}{\partial y \partial x} = -6$$

$$\frac{\partial f}{\partial y} = -6x - 6y = 0 \qquad x + y = 0, \; \boxed{y = -x}$$

$$\frac{\partial^2 f}{\partial y^2} = -6$$

$$x^2 - y = 0, \quad x^2 + x = 0$$

$$x(x+1) = 0 \qquad x = 0 \text{ or } x = -1.$$

Critical points: $x = 0, \; y = 0 \quad (0,0)$

$x = -1, \; y = 1 \quad (-1,1)$

Step 2 : $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$

$$\frac{\partial^2 f}{\partial x \partial y} = f_{xy} = -6$$

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = \underline{\underline{12x}}$$

$$\frac{\partial^2 f}{\partial y^2} = -6 = f_{yy}$$

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

$$D = -72x - 36$$

when $x=0$, $D = -36$ So the point $(0,0)$ is a Saddle point

when $x=-1$, $D = 72 - 36 = 36 > 0$

$f_{yy} = -6 < 0$

So the point $(-1, 1)$

is a local maximum.

relative

25| $f(x,y) = 4x^2 + y^2$

on $xy = 1$

Solution: Use Lagrange multipliers

$g(x,y) = xy$

$f(x,y) = 4x^2 + y^2$

$\nabla f = \lambda \nabla g$

$\nabla f = (8x\vec{i} + 2y\vec{j}) = \lambda(y\vec{i} + x\vec{j})$

$$8x = \lambda y \qquad xy = 1$$

$$2y = \lambda x$$

$$\frac{8x}{2y} = \frac{\cancel{\lambda} y}{\cancel{\lambda} x}$$

$$4\frac{x}{y} = \frac{y}{x} \qquad \boxed{4x^2 = y^2}$$

But $xy = 1$ $\qquad y = 1/x$

$$4x^2 = 1/x^2 \; ; \quad 4x^4 = 1$$

$$x^4 = 1/4 \qquad x = \pm \frac{1}{(4)^{1/4}}.$$

$$x = \frac{1}{(4)^{1/4}}, \quad y = 4^{1/4}$$

$$x = -1/(4)^{1/4}, \quad y = -4^{1/4}.$$