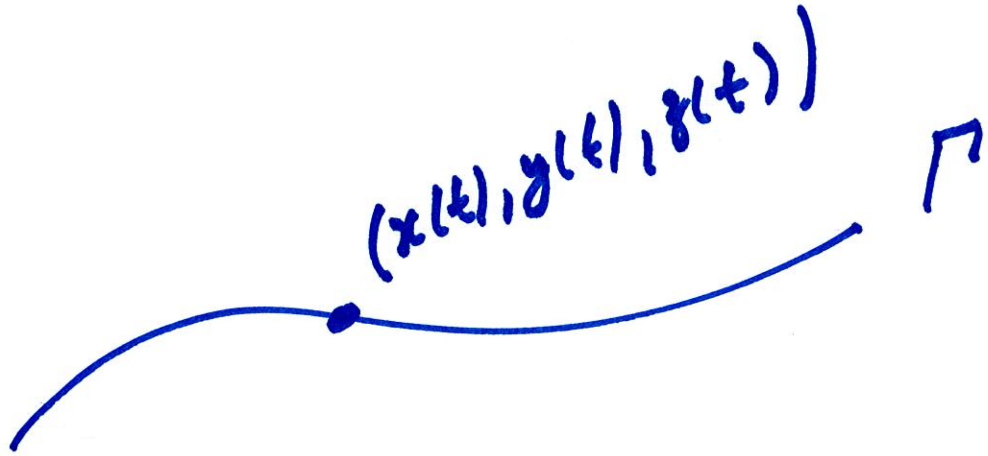
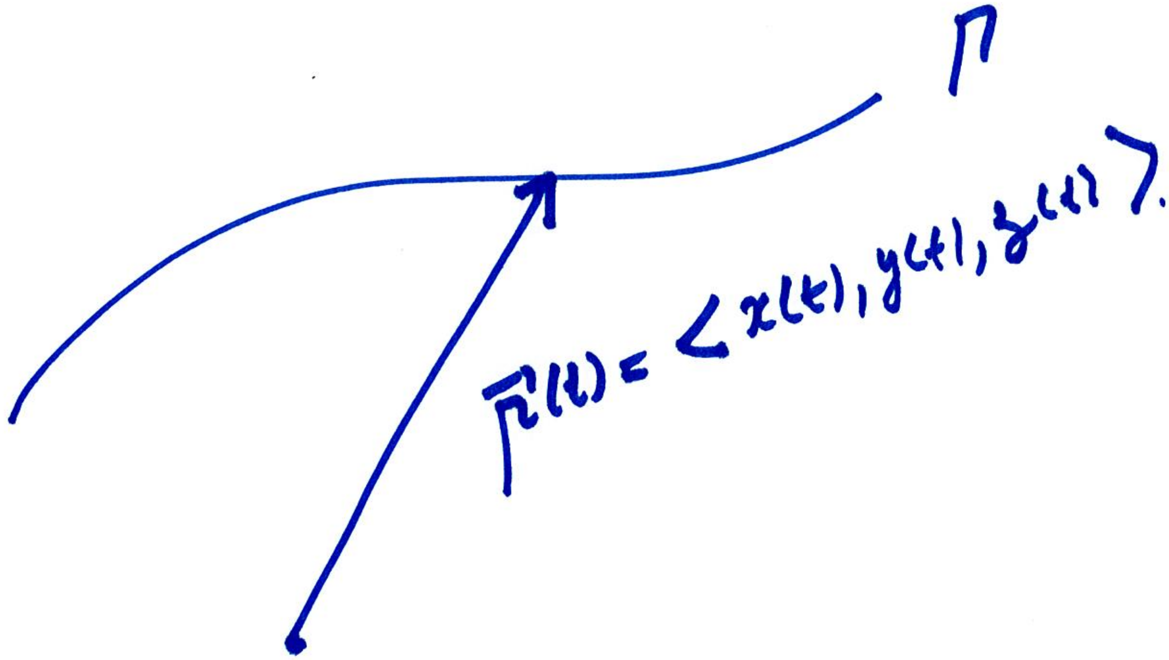
Lesson 7 Section 13.3



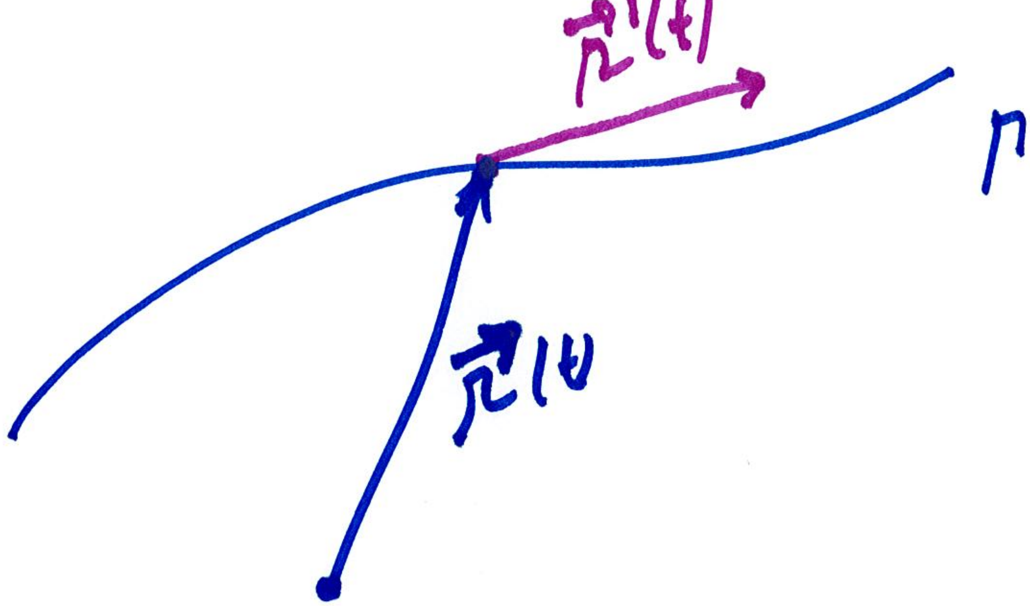
Γ is a curve with vector equation $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

Any point (x, y, z) on Γ is given by parametric equations

$x = x(t)$, $y = y(t)$ and $z = z(t)$.



Use the vector equation and parametric equations to understand properties of Γ.

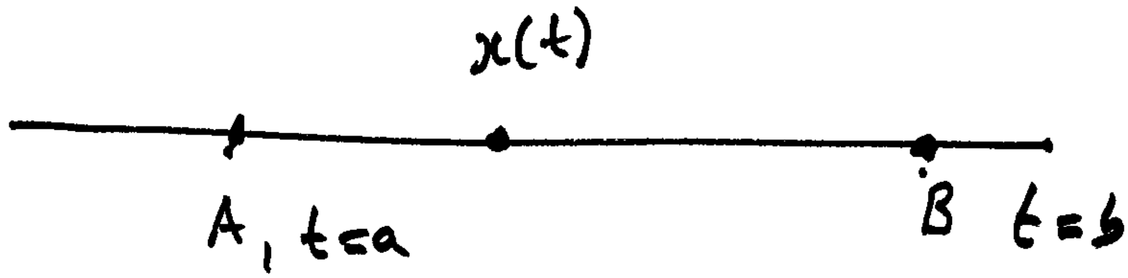


$\vec{r}'(t)$ is tangent to Γ at $\vec{r}(t)$

$|\vec{r}'(t)| =$ speed of movement.

<u>Example:</u>

$$\vec{r}(t) = f(t)\,\vec{v}.$$



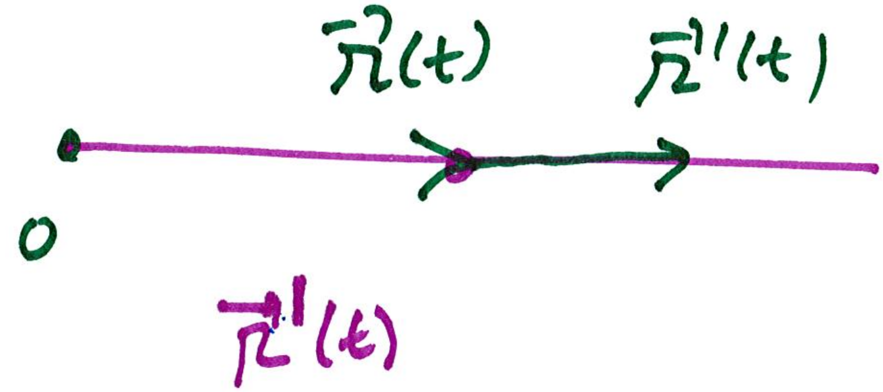
$$\text{Speed} = |x'(t)|$$

$$L = \int_a^b |x'(t)|\, dt$$

If one travels with constant speed

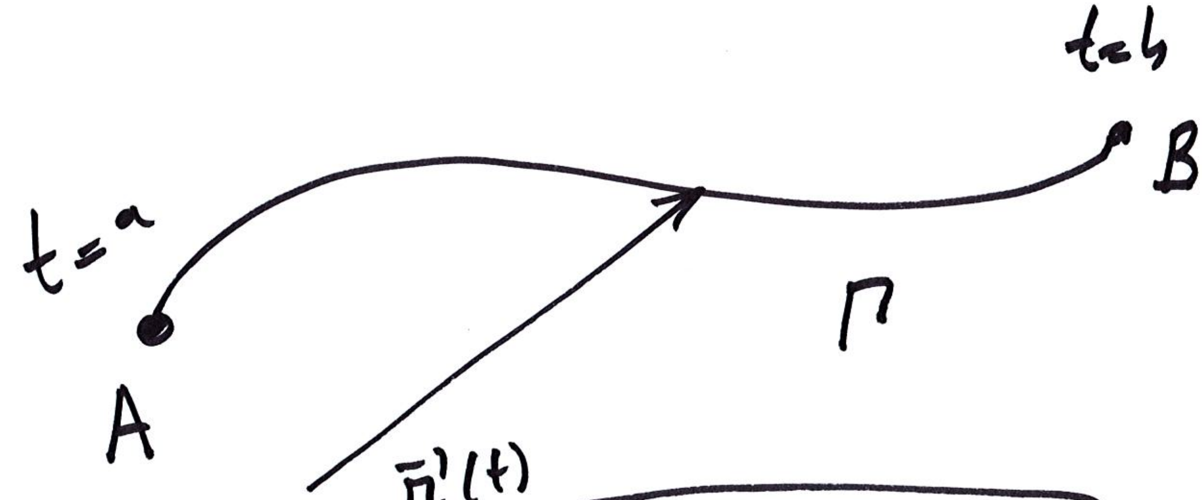
$$x'(t) = 10$$

$$L = \int_a^b 10\, dt = \underline{10\,(b-a)}.$$



$$\text{Speed} = \frac{df}{dt}$$

Distance & Arc Length



$$\boxed{L = \int_a^b |\vec{r}\,'(t)|\, dt.}$$

Curves in 2-dimensions

$$\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j}$$

$$L = \int_a^b |\vec{r}\,'(t)|\,dt$$

$$= \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\,dt.$$

In 3-d.

$$\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$$

$$a \le t \le b.$$

$$L = \int_a^b \sqrt{(f'(t))^2 + (g(t))^2 + (h'(t))^2}\, dt.$$

<u>Example:</u> Find the length of the curve with vector equation

$$\vec{r}(t) = t\cos t\,\vec{i} + t\sin t\,\vec{j} + \frac{2}{3}\sqrt{2}\, t^{3/2}\,\vec{k}$$

$$1 \le t \le 2$$

$$L = \int_1^2 |\vec{r}\,'(t)|\, dt.$$

$$\vec{r}\,'(t) = (\cos t - t\sin t)\,\vec{i} + (\sin t + t\cos t)\,\vec{j} + \sqrt{2}\,t^{1/2}.$$

$$|\vec{r}\,'(t)|^2 = (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 2t.$$

$$= \underbrace{\cos^2 t} - \cancel{2t\sin t\cos t} + \underbrace{t^2\sin^2 t} + \underbrace{\sin^2 t} + \cancel{2t\sin t\cos t} + \underbrace{t^2\cos^2 t} + 2t.$$

$$= 1 + t^2 + 2t = (1+t)^2.$$

$$\boxed{|\vec{r}\,'(t)| = 1+t.}$$

$$L = \int_1^2 |\vec{r}\,'(t)|\,dt$$

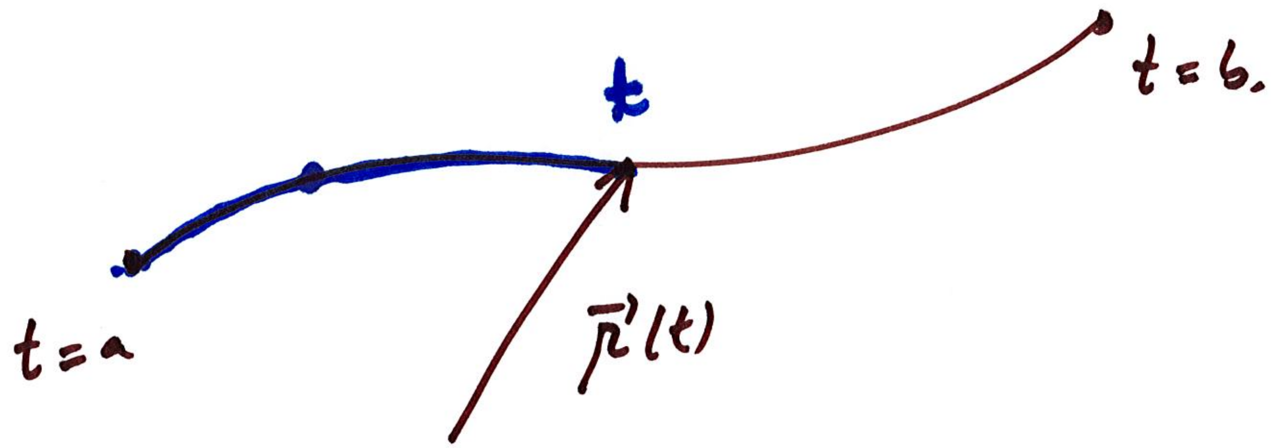
$$= \int_1^2 (1+t)\,dt$$

$$= \left(t + \frac{t^2}{2}\right)\Big|_1^2 =$$

$$= \left(2+\frac{4}{2}\right) - \left(1+\frac{1}{2}\right)$$

$$= 4 - \frac{3}{2} = \frac{5}{2}.$$

The Arc-Length Function



$$S(t) = \int_a^t |\vec{r}\,'(u)|\, du$$

In the previous example:

$$S(t) = \int_1^t |\vec{r}\,'(u)|\, du = \int_1^t (1+u)\, du$$

$$S(t) = \left(u + \frac{u^2}{2} \right) \Big|_1^t$$

$$= \left(t + \frac{t^2}{2} \right) - \left(1 + \frac{1}{2} \right)$$

$$= \frac{t^2}{2} + t - \frac{3}{2}.$$

$$\boxed{S(t) = \frac{t^2}{2} + t - \frac{3}{2}}$$

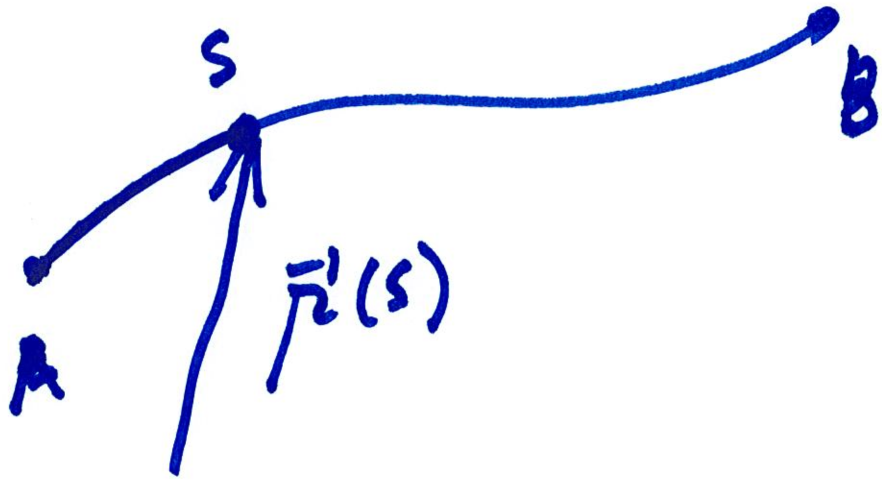
<u>when $t = 2$</u>: $S(2) = 4 - \frac{3}{2} = \frac{5}{2}$.

Arc-Length Parametrization

Γ is described by a vector equation $\vec{r}(t)$.

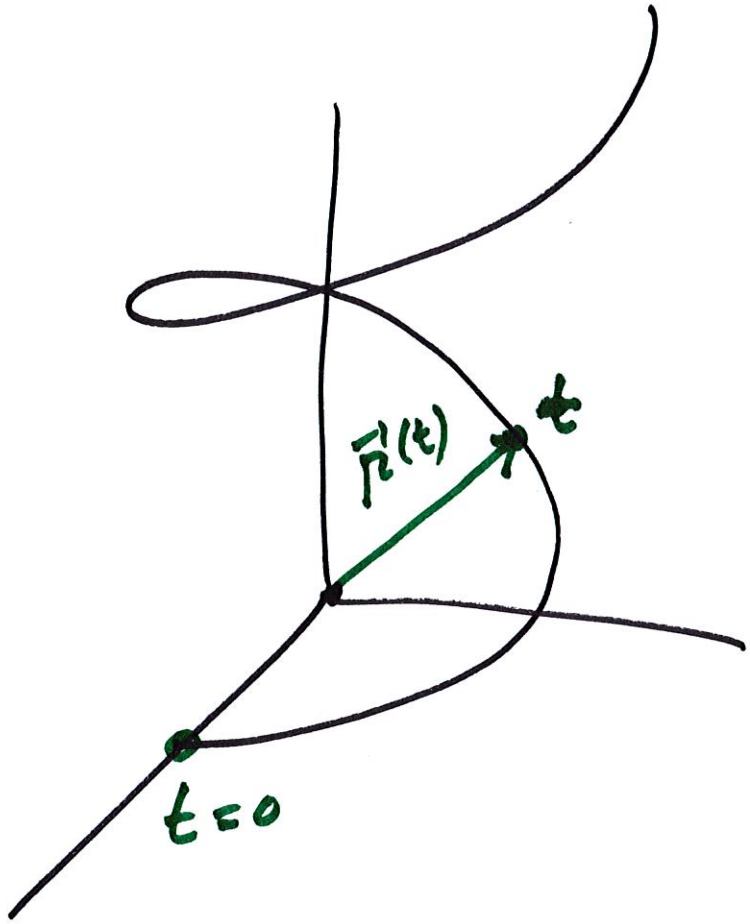
We choose the parameter t

$$|\vec{r}\,'(t)| = 1.$$



s = Length of the curve from A to $\vec{r}(s)$.

<u>Example:</u> $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$.



Express $\vec{r}$ in terms of the arc-length parameter. s.

$$s = \int_0^t |\vec{r}'(u)|\,du.$$

$$\vec{r}'(u) = -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}.$$

$$|\vec{r}'(u)| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}.$$

$$s(t) = \int_0^{t} \sqrt{2}\, du = \sqrt{2}\, t.$$

$$s = \sqrt{2}\, t$$

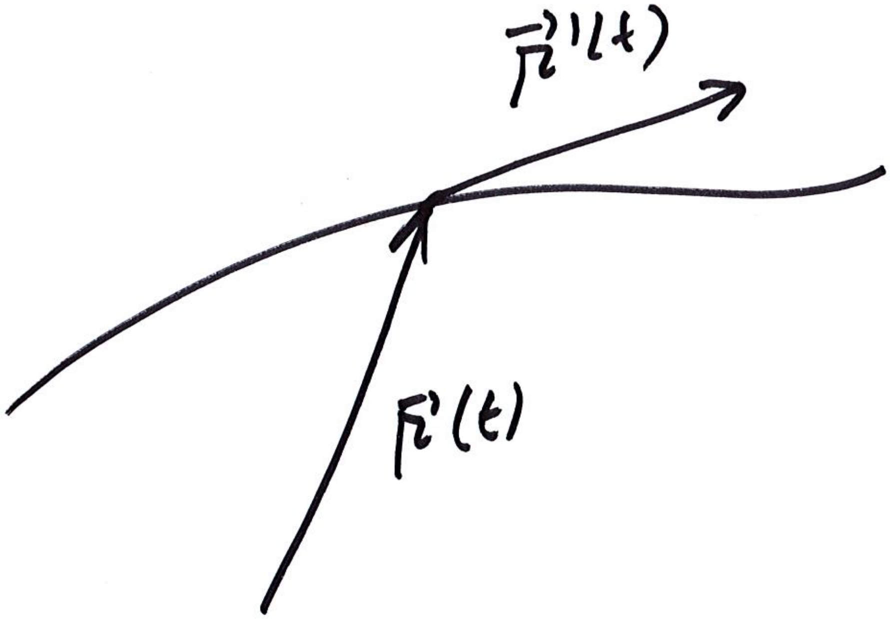
$$\boxed{t = s/\sqrt{2}}$$

$$\vec{r}(t) = \cos t\, \vec{i} + \sin t\, \vec{j} + t\, \vec{k}$$

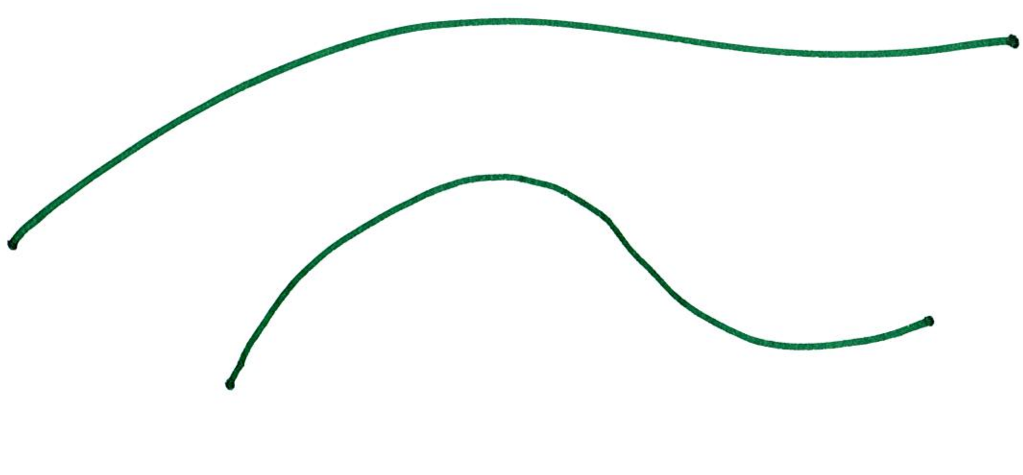
In terms of s:

$$\boxed{\vec{r}(s) = \cos\left(s/\sqrt{2}\right)\vec{i} + \sin\left(s/\sqrt{2}\right)\vec{j} + s/\sqrt{2}\; \vec{k}}$$

Unit tangent Vector



$$\boxed{\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}}$$



Curvature.

$$k = \left| \frac{d\vec{T}}{ds} \right|$$

Simpler formula:

$$k = \frac{\left| \frac{dT}{dt} \right|}{\left| \frac{dr}{dt} \right|}$$