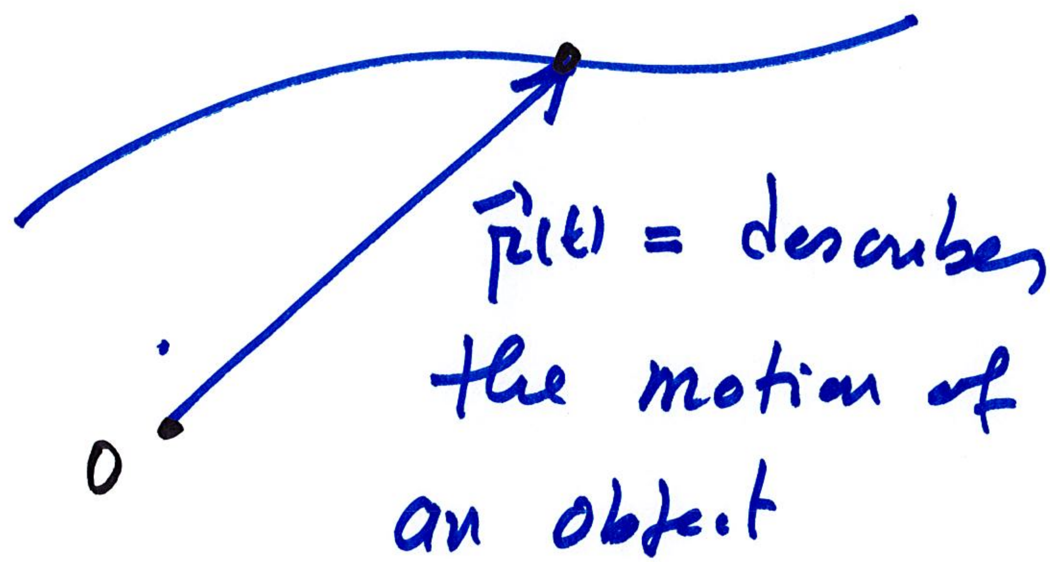
Lesson 8   Section 13.4

Motion in Space



$\vec{r}(t)$ = Position of the object at time t.

$$\vec{V}(t) = \frac{d}{dt}\vec{r}(t)$$

$$\text{Speed} = |\vec{V}(t)| = |\vec{r}\,'(t)|.$$

<u>Acceleration</u>:

$$\vec{a}(t) = \frac{d\vec{V}}{dt} = \vec{r}\,''(t)$$

<u>Examples:</u>

$$\vec{r}(t) = e^{t}\,\vec{i} + e^{-t}\,\vec{j} + t^{2}\,\vec{k}.$$

Find $\vec{V}(t)$; $\vec{a}(t)$ and the Speed.

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = e^{t}\vec{i} - e^{-t}\vec{j} + 2t\vec{k}.$$

$$\text{Speed} = |\vec{v}(t)| = \sqrt{e^{2t} + e^{-2t} + 4t^{2}}.$$

Accelerahm.

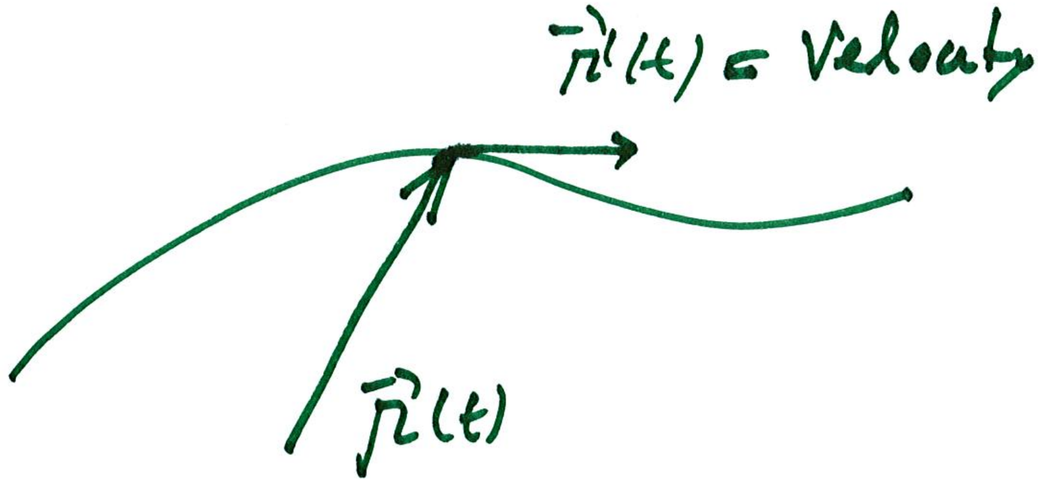
$$\vec{a}(t) = e^{t}\vec{i} + e^{-t}\vec{j} + 2\vec{k}.$$

Remark:

$$|\vec{a}(t)| = \left|\frac{d}{dt}\vec{v}(t)\right| \neq \frac{d}{dt}|\vec{v}(t)|$$

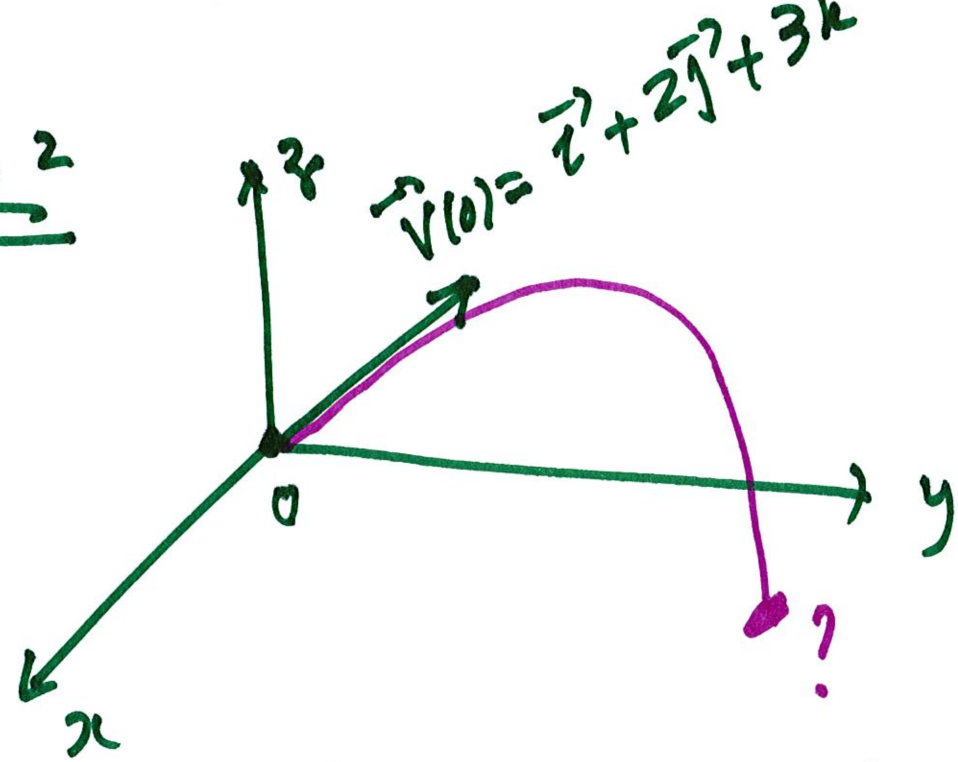
$$|\vec{a}(t)| = \sqrt{e^{2t} + e^{-2t} + 4}.$$

$$\frac{d}{dt}|\vec{v}(t)| = \frac{1}{2\sqrt{e^{2t} + e^{-2t} + 4t^{2}}}\left(2e^{2t} - 2e^{-2t} + 8t\right).$$



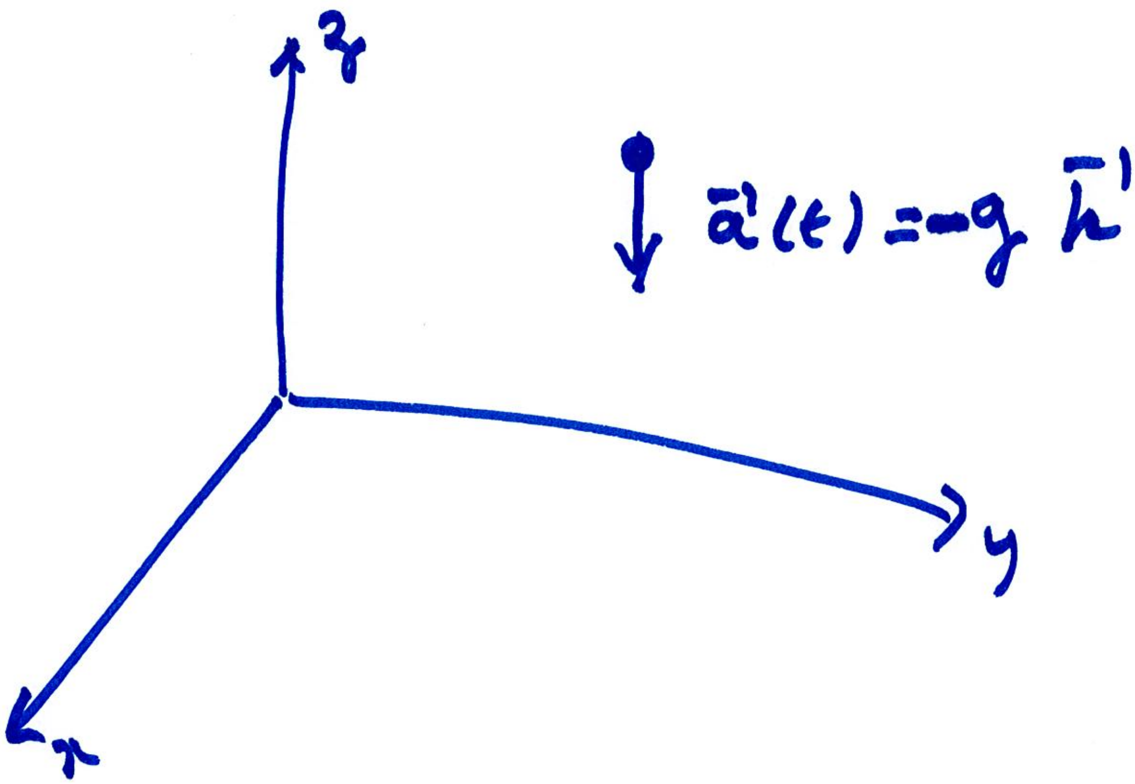
$\vec{r}''(t)$

Example 2



An object is thrown from the point 0 with initial velocity $\vec{v}(0) = \vec{i} + 2\vec{j} + 3\vec{k}$.

Question: where is it going to land?



$$\vec{a}(t) = \frac{d\vec{v}}{dt}.$$

$$\vec{v}(t) = \int \vec{a}(t)\,dt + \vec{v}(0)$$

$$\frac{d\vec{v}}{dt} = 0\,\vec{i} + 0\,\vec{j} - g\,\vec{k}.$$

$$V(t) = c_1 \vec{i} + c_2 \vec{j} - (gt + c_3)\vec{k}.$$

when $t = 0$

$$\vec{V}(0) = c_1 \vec{i} + \vec{c_2}\, \vec{j} - c_3 \vec{k}.$$

$$= \vec{i} + 2\vec{j} + 3\vec{k}$$

$$c_1 = 1, \quad c_2 = 2, \quad c_3 = -3.$$

$$\boxed{\vec{V}(t) = \vec{i} + 2\vec{j} - (gt - 3)\vec{k}}$$

$$\frac{d\vec{r}}{dt} = \vec{V}(t) = \vec{i} + 2\vec{j} - (gt - 3)\vec{k}.$$

$$\vec{r}(t) = (t + d_1)\,\vec{i} + (2t + d_2)\,\vec{j} - \left(\frac{1}{2} g t^2 - 3t + d_3\right)\vec{k}.$$

$$\vec{r}(0) = 0. = d_1\,\vec{i} + d_2\,\vec{j} - d_3\,\vec{k}$$

$$d_1 = d_2 = d_3 = 0$$

$$\boxed{\vec{r}(t) = t\,\vec{i} + 2t\,\vec{j} - \left(\underbrace{\frac{1}{2} g t^2 - 3t}_{}\right)\vec{k}}$$

The object lands when ...

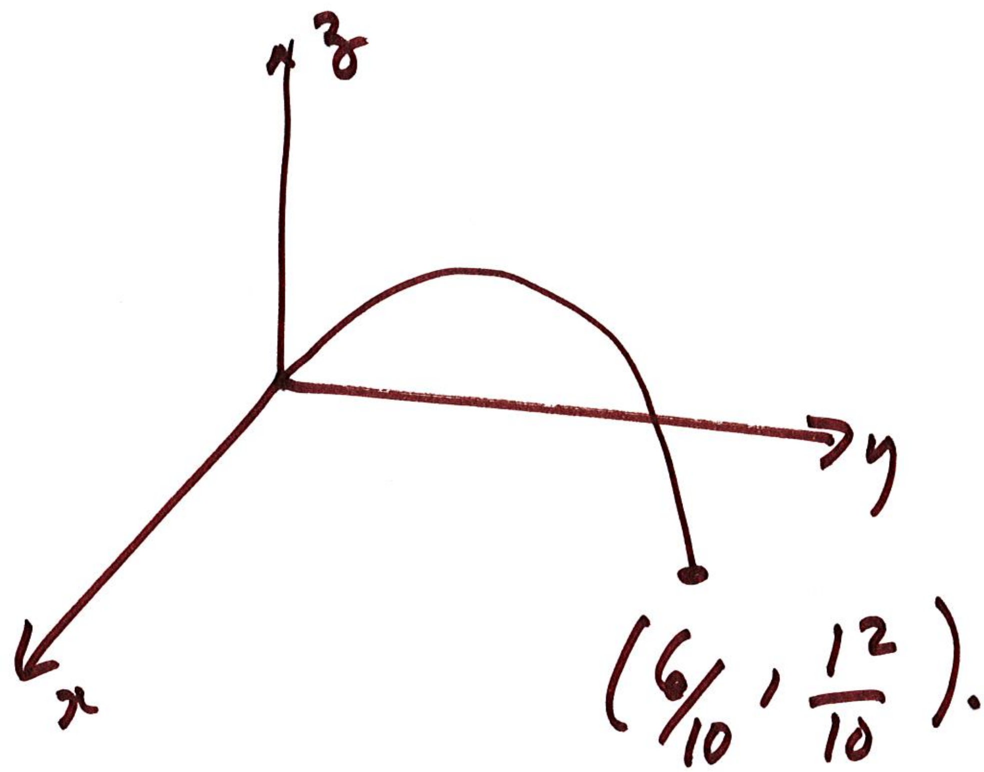
$$\frac{1}{2} g t^2 - 3t = 0$$

$\underline{t = 0}$. , $\frac{1}{2} g t - 3 = 0$, $g t = 6$

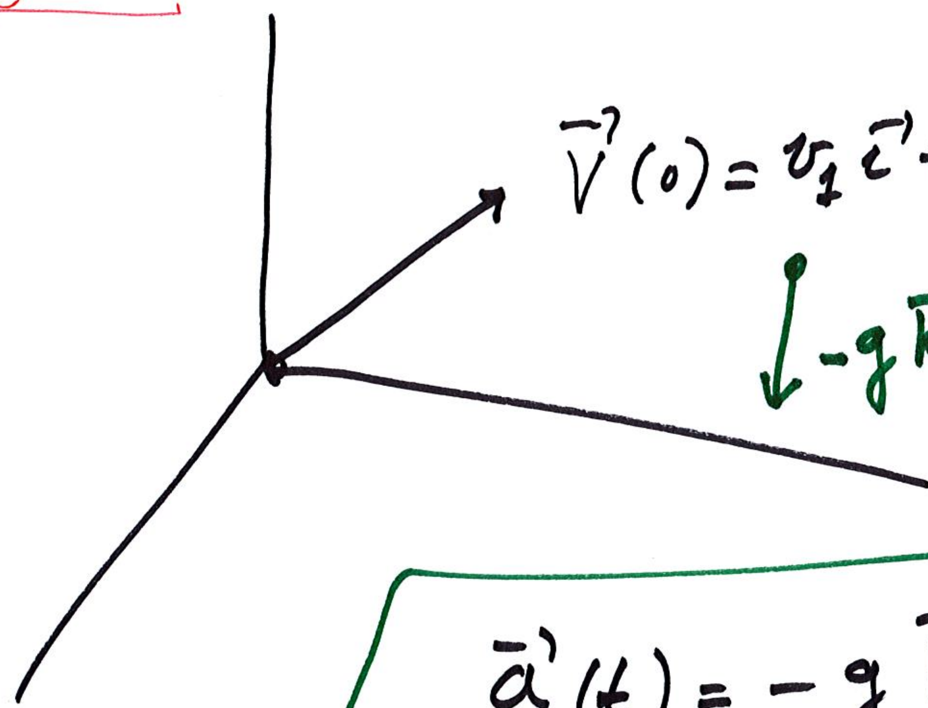
$$\boxed{t = 6/g}$$

$g = 10\ m/s^2$.

$$\boxed{\vec{r}\left(6/10\right) = \frac{6}{10}\,\vec{i} + \frac{12}{10}\,\vec{j}}$$



In general



$$\vec{V}(0) = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$$

$$-g\vec{k}$$

$$\boxed{\vec{a}(t) = -g\vec{k}.}$$

$$V(t) = c_1 \vec{i} + c_2 \vec{j} - (gt + c_3)\vec{k}.$$

$$\vec{V}(t) = v_1 \vec{i} + v_2 \vec{j} - (gt - v_3)\vec{k}.$$

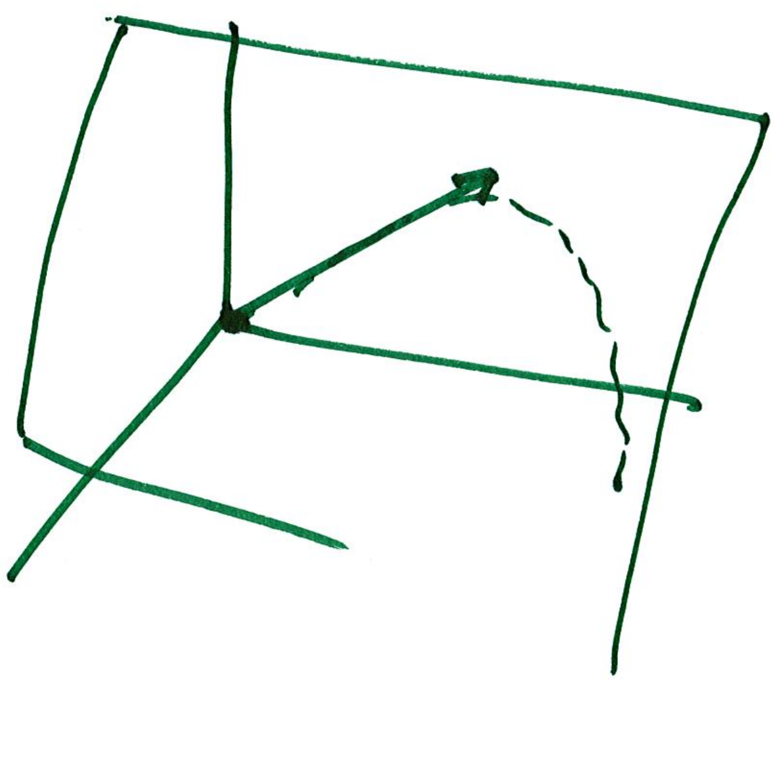
$$\vec{r}(t) = \boxed{v_1 t \vec{i} + v_2 t \vec{j}} - \left(\frac{gt^2}{2} - v_3 t\right)\vec{k}.$$

$$x = v_1 t, \quad y = v_2 t.$$

$$v_1 \neq 0 \qquad t = x/v_1$$

The motion is restricted to the plane

$$\boxed{y = \frac{v_2}{v_1} x}$$



Example: the position of a particle is given by

$$\vec{r}(t) = t\cos t\,\vec{i} + t\sin t\,\vec{j} + \left(\frac{t^3}{3} - 2t^2\right)\vec{k}.$$

Find the time when the speed is a minimum.

Solution:

Step 1: Find $|\vec{r}\,'(t)|$

Step 2: Find t such that

$$\frac{d}{dt}|\vec{r}\,'(t)| = 0.$$

Step 3: Make sure it is a minimum.

Step 1: $\vec{r}\,'(t) = (\cos t - t\sin t)\,\vec{i}$

$+ (\sin t + t\cos t)\,\vec{j}$

$+ (t^2 - 4t)\,\vec{k}.$

$$|\vec{r}\,'(t)|^2 = (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + (t^2-4t)^2$$

$$= \underbrace{\cos^2 t} - \cancel{2t\sin t\cos t} + \underbrace{t^2\sin^2 t} + \underbrace{\sin^2 t} + \underbrace{t^2\cos^2 t} + \cancel{2t\sin t\cos t} + (t^2-4t)^2$$

$$= 1 + t^2 + (t^2-4t)^2.$$

$$|\vec{r}\,'(t)| = \sqrt{1+t^2+(t^2-4t)^2}.$$

$$\frac{d}{dt}|\vec{r}\,'(t)| = \frac{1}{2\sqrt{1+t^2+(t^2-4t)^2}}\left(2t+2(t^2-4t)(2t-4)\right)$$

We want

$$\cancel{2}t + \cancel{2}(t^2-4t)(2t-4) = 0$$

$$t + t(t-4)(2t-4) = 0$$

<u>Solution 1</u> : <u>$t = 0$</u>

If $t \neq 0$

$$1 + (t-4)(2t-4) = 0$$

$$= 1 + 2t^2 - 4t - 8t + 16$$

$$\boxed{2t^2 - 12t + 17 = 0}$$

$$t = \frac{12 \pm \sqrt{144 - 8 \times 17}}{2}$$

$$t = \frac{12 \pm \sqrt{144 - 136}}{2}$$

$$t = \frac{12 \pm \sqrt{8}}{2}.$$

$$t_1 = \frac{12 + \sqrt{8}}{2}$$

$$t_2 = \frac{12 - \sqrt{8}}{2}.$$