MA 262 2/5/2016

Lesson 11: The Inverse of a Square Matrix Section 2.6

$$A = [a_{jk}]_{n \times n}$$

The inverse of A is the matrix A^{-1}

such that

$$A A^{-1} = A^{-1} A = I_n = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

Example: $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$; $A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

$$A A^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} A = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ +1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Not all square matrices have inverses.

Ex, $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ does not have an inverse.

$$A \quad \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} a + 2c &= 1 \\ a + 2c &= 0. \end{aligned}$$

Oops!

How do we compute the inverse of a matrix or decide if it does not have one?

$$A \cdot \begin{array}{c} \begin{array}{cccc} \downarrow B_1 & \downarrow B_2 & & \downarrow B_n \end{array} \\ \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ & & \ddots & 0 \\ 0 & & & 1 \end{bmatrix}$$

We have to simultaneously solve

the systems.

$$A B_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \quad A B_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots \quad A B_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Augmented matrix

$$\left[\begin{array}{c|c} \mathbf{A} & \begin{matrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right] \xrightarrow[\text{echelon}]{\text{Reduced Row}} \left[\begin{array}{cccc|c} 1 & & & & \\ & 1 & & & B_1 \\ & & \ddots & & \\ & & & 1 & \end{array}\right]$$

So we ~~need~~. Work

$$\left[\begin{array}{c|c} A & Id \end{array}\right] \xrightarrow[\text{Echelon}]{\text{Reduced Row}}$$

$$\left[\begin{array}{c|c} Id & A^{-1} \end{array}\right]$$

2-1

Example: $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, Find A^{-1}

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We want
$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 0 \end{bmatrix} \sim \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 2 & -2 \end{array}\right]$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \end{bmatrix}$$

So, $b_{11} = 4$
$b_{21} = -1.$

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 5 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 2 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & 1/2 \end{bmatrix}$$

Instead of doing this twice we write.

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 2 & -2 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -1 & 1/2 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & +4 & -3/2 \\ 0 & 1 & -1 & 1/2 \end{array}\right]$$

Important Facts

(1) A has an inverse if and only if $\text{Rank } A = n$.

(2) If A has an inverse, and

$$Ax = b$$

$$A^{-1}Ax = A^{-1}b$$

$$\boxed{x = A^{-1}b}$$

(3) <u>Example</u>: Find the inverse of

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 2 & 1 & 1 \end{bmatrix}$$

~~and use it to solve the system.~~

$$\cancel{Ax = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad -2R_2+R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & +1 & +3 & +2 & 0 & -1 \end{array}\right] \quad -R_2+R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \end{array}\right]$$

No Inverse.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix}, \quad \text{Find } A^{-1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & +1 & 0 & +2 & 0 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -2/3 & 1/3 & 1/3 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & -2/3 & 1/3 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -2/3 & 1/3 & 1/3 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} -1/3 & -2/3 & 1/3 \\ 2 & 0 & -1 \\ -2/3 & 1/3 & 1/3 \end{bmatrix}}$$