MA 262 : 2/8/2016

Lesson 12 : Section 3.2 Properties of determinants

$A = \left[a_{jk} \right]_{n \times n}$ a square matrix.

Definition If A is an upper triangular matrix or lower

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & & a_{2n} \\ & & \ddots & \vdots \\ & 0 & & a_{nn} \end{bmatrix}$$

$$\det A = a_{11}\, a_{22} \cdots a_{nn}.$$

Example. $\det \begin{bmatrix} 2 & 6 & 500 \\ 0 & -3 & \pi \\ 0 & 0 & 2 \end{bmatrix} = -12.$

Determinant of an arbitrary square matrix: A is calculated by reducing it to row-echelon form and observing the following properties:

P.1. If B is the matrix obtained by permuting two rows of A, $\det B = -\det A$

P.2: If B is the matrix obtained by multiplying one row of A by a number k,

$$\det B = k \det A$$

P.3: If B is the matrix obtained by multiplying one row of A by k and adding the result to another row of A

$$\det B = \det A.$$

Ex: Compute $\det \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 10 \\ 2 & 5 & 8 \end{bmatrix} =$

$$= \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= -3.$$

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $\det A = -6.$

Find $\underline{\det B}$: $B = \begin{bmatrix} d & e & f \\ -3a & -3b & -3c \\ g-4d & h-4e & i-4f \end{bmatrix}$

(1) multiply $R1 \times -3 = (-3) \cdot 6 = -18$

(2) Permute R_1 with $R_2 = 18$

(4) $R_3 - 4R_2 = \det A = \underline{\underline{18}}$

~~Ex: Find all x such that~~

Ex: Compute

$$\det \begin{bmatrix} -1 & 2 & 3 \\ 5 & -2 & 1 \\ 8 & -2 & 5 \end{bmatrix} =$$

$$= \det \begin{bmatrix} -1 & 2 & 3 \\ 0 & 8 & 6 \\ 0 & 14 & 29 \end{bmatrix} = \frac{1}{8} \det \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6/8 \\ 0 & 14 & 29 \end{bmatrix}$$

$$= \frac{1}{8} \det \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6/8 \\ 0 & 0 & 29 - 14 \times \frac{6}{8} \end{bmatrix} = -\frac{37}{2}.$$

$$29 - \frac{21}{2} = \frac{37}{2}$$

Properties of determinant

$\det A = \det A^T$ This is easy to verify for triangular matrices.

Properties P. 1, P. 2 and P. 3 also hold for column operations.

~~Further Properties: det(AB) = det A . det B~~ ⑤

Without expanding the determinant.
Find all values of x for which

$$\det \begin{bmatrix} 1 & -1 & x \\ 2 & 1 & x^2 \\ 4 & -1 & x^3 \end{bmatrix} =$$

$$= x \det \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & x \\ 4 & -1 & x^2 \end{bmatrix} =$$

$$= x \det \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & x-2 \\ 0 & 3 & x^2-4 \end{bmatrix}$$

$$= x \det \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & x-2 \\ 0 & 0 & x^2-4-x+2 \end{bmatrix}$$

$$3x\,(x^2-x-2) = 3x\,(x-2)(x+1)$$

$x=0, \quad x=2, \quad x=-1.$