MA262 2/10/2016

Lesson 13 Section 3.3 Cofactor Expansions

$$A = [a_{jk}]_{n \times n}$$

Minors:

M_{jk} = determinant of the matrix obtained from A by deleting the i-th row and j-th column.

Ex: $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 8 \\ 1 & 4 & 2 \end{bmatrix}$

$M_{11} = \det \begin{bmatrix} 7 & 8 \\ 4 & 2 \end{bmatrix}$; $M_{23} = \det \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$

$= 14 - 32 = -18$ $= 2$.

Cofactor: $C_{jk} = (-1)^{j+h} M_{jk}$

(signed minor)

$$C_{11} = (-1)^{1+1} M_{11} = -18$$

$$C_{23} = (-1)^{2+3} M_{23} = -2.$$

The cofactor expansion theorem

$$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \dots + a_{in} C_{in}.$$

$$= a_{1j} C_{1j} + a_{2j} C_{2j} + \dots + a_{nj} C_{nj}.$$

Example:

Evaluate: $\det \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 6 \\ 2 & 1 & 4 \end{bmatrix}$

Row 1: $(-1)^{1+1} \cdot 2 \cdot \det \begin{bmatrix} 0 & 6 \\ 1 & 4 \end{bmatrix}$

$+ (-1)^{1+2} \cdot 1 \cdot \det \begin{bmatrix} 1 & 6 \\ 2 & 4 \end{bmatrix}$

$+ (-1)^{1+3} \cdot 3 \det \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$= \cancel{-12 = (4-12)}$

$= 2(0-6) - 1(4-12) + 3 \cdot (1-0)$

$= -12 + 8 + 3 = -1$

Column 3:

$(-1)^{1+3} \cdot 3 \det \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$+ (-1)^{2+3} \cdot 6 \det \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$

$+ (-1)^{3+3} \cdot 4 \det \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

$= 3 \cdot (1-0) - 6(2-2) + 4(0-1)$

$= 3 - 4 = -1.$

Compute:

$$\det \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 3 \\ 0 & 2 & -1 & 6 \\ 1 & 3 & -2 & 4 \end{bmatrix}$$

$$= - \det \begin{bmatrix} 1 & 4 & -2 & 3 \\ 2 & -1 & 3 & 1 \\ 0 & 2 & -1 & 0 \\ 1 & 3 & -2 & 4 \end{bmatrix}$$

$$= - \det \begin{bmatrix} 1 & 4 & -2 & 3 \\ 0 & -9 & 7 & -5 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$= - \det \begin{bmatrix} -9 & 7 & -5 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \det \begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & 0 \\ -9 & 7 & -5 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -9 & 7 & -5 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 7 & 4 \end{bmatrix}$$

$$= -\det \begin{bmatrix} -1 & -2 \\ 7 & 4 \end{bmatrix}$$

$$= -\left(-4 + 14\right) = -10.$$