Lesson 14 : 2/12/2016

The adjoint method for A^{-1}.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

M_c = The matrix of cofactors

$$M_c = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

$$\operatorname{adj}(A) = M_c^T$$

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$M_c = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$$

$$\mathrm{adj}\, A = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Cramer's Rule

$$Ax = b. \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & & a_{3n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} ; \; b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$B_1 = \begin{bmatrix} b_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ b_2 & a_{22} & a_{23} & & a_{2n} \\ \vdots & \vdots & & & \\ b_n & a_{n2} & a_{n3} & & a_{nn} \end{bmatrix}$$

$$B_k = \begin{bmatrix} a_{11} & \cdots & a_{1k-1} & b_1 & a_{1k+1} & \cdots & a_{1n} \\ a_{21} & & a_{2k-1} & b_2 & a_{2k+1} & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & \\ a_{n1} & & a_{nk-1} & b_n & a_{nk+1} & & a_{nn} \end{bmatrix}$$

Replace the k-th column with the vector b.

$$\boxed{x_k = \frac{\det B_k}{\det A}}$$

Example: Solve

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + x_2 - x_3 &= 0 \\ x_1 \quad\quad - x_3 &= 1 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} ; \quad B_2 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$