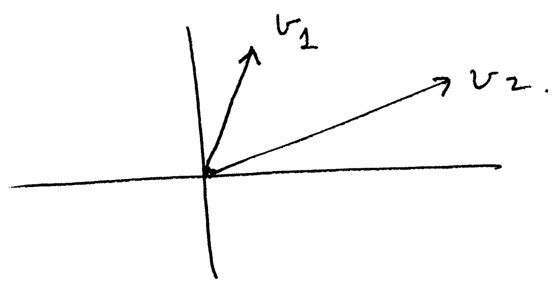
Linear Dependence and Linear Independence.



Any vector $v \in \mathbb{R}^2$ is of the form

$v = c_1 v_1 + c_2 v_2$.

Ex. (1) $v_1 = (1,0)$; $v_2 = (0,1)$ $v = (x,y)$

$v = (x,y) = x(1,0) + y(0,1) = x v_1 + y v_2$.

(2) We say that v, v_1 and v_2 are **linearly dependent** because v can be written as a linear combination of v_1 and v_2.

Ex. $v_1 = (1,1,2)$; $v_2 = (0,1,3)$

are linearly independent because one is not a multiple of the other

Question: Are $v_1 = (1, 2, 1)$, $v_2 = (0, 1, 3)$ and $v_3 = (1, 0, 1)$ linearly dependent?

In other words, are there x_1, x_2, x_3 such that

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\det \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 3 & 0 \end{bmatrix} = 6 \neq 0$$

Only solution is $x_1 = x_3 = 0.$

v_1, v_2 and v_3 are <u>linearly independent</u>.

V is a vector space. and

$v_1, \ldots, v_n \in V$.

We say that $v_1, \ldots, v_n$ are linearly dependent if there exist scalars $c_1, c_2, \ldots, c_n$ all non-zero

such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

On the other hand $v_1, v_2, \ldots, v_n$ are linearly independent if and only if the only values of $c_1, c_2, \ldots, c_k$

such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

are $c_1 = c_2 = \cdots = c_k = 0$

Ex: $\{(1,-1,2,3);\ (2,-1,1,-1);\ (-1,1,1,1)\}$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -5 & 4 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \quad \boxed{\begin{array}{l} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{array}}$$

Linear Independence of function

$\{ f_1, f_2, \dots, f_n \}$ are linearly independent

on an interval I if and only if

$$c_1 f_1 + c_2 f_2 + \dots + c_k f_k = 0$$

implies that $c_1 = c_2 = \dots = c_k = 0$

$$W[f_1, f_2, \dots, f_k](x) = \det \begin{bmatrix} f_1(x) & f_2(x) & \dots & f_k(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ \vdots & & & \\ f_1^{(k-1)}(x) & f_2^{(k-1)}(x) & \dots & f_k^{(k-1)} \end{bmatrix}$$

Wronskian

If $W[f_1, \dots f_k](x_0) \neq 0$

Linear Independence

for some $x_0 \in I$.

$f_1, \dots, f_n$ linearly independent

but $W(x) = 0$ and $f_1, \dots, f_n$ be l.i.

Ex :

$$f_1(x) = \sin x \,; \quad f_2(x) = \cos x$$

$$f_3(x) = \tan x$$

$$W = \begin{bmatrix} \sin x & \cos x & \tan x \\ \cos x & -\sin x & \sec^2 x \\ -\sin x & -\cos x & 2\sec^2 x \tan x \end{bmatrix}$$