Bases and Dimension.

Example: $V_1 = (1, 2, 1)$; $V_2 = (0, 1, 1)$
and $V_3 = (0, 1, 0)$.

What is the span of $\{V_1, V_2, V_3\}$?

$$V = x_1 V_1 + x_2 V_2 + x_3 V_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \det A = -1$$

A is invertible, the system has a solution for any V.

We say that $\{V_1, V_2, V_3\}$ form a basis of $\mathbb{R}^3$.

In general we say that a set of vectors $\{V_1, V_2, \dots, V_K\}$ in a vector space V is called a basis for V if

(1) The vectors are linearly independent.

(2) They span V.

<u>Example:</u> $M_2(\mathbb{R})$ 2x2 matrices over $\mathbb{R}$

~~Met~~ $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} =$

$= a_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$+ a_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ form a basis of $M_2(\mathbb{R})$

Symmetric matrices in $M_3(\mathbb{R})$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} =$$

$$= a_{11}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ a_{13}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + a_{23}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$+ a_{22}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{33}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ok

Find the dimension of the null space of the matrix (5)

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & 3 & 7 & 10 \\ 3 & 4 & 9 & 13 \end{bmatrix}$$

$$Ax = 0 \; : \; \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & 4 & 0 \\ 2 & 3 & 7 & 10 & 0 \\ 3 & 4 & 9 & 13 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & 4 & 0 \\ 0 & 1 & 3 & 4 & 0 \\ 0 & 1 & 3 & 4 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_2 + 3x_3 + 4x_4 = 0 \qquad x_2 = 3x_3 + 4x_4$$

$$x_1 - x_3 - x_4 = 0 \qquad x_1 = x_3 + x_4$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + x_4 \\ 3x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

<u>Bans</u>: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} ; \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\{$