MA 262 1/15/16 Lesson 3

Section 1.6 1st order LDE

$$\frac{dy}{dx} + p(x)y = f(x)$$

Easy case: $p(x) = 0$; $\frac{dy}{dx} = f(x)$ and

$$y(x) = \int f(x)\,dx + C.$$

General Case Reduce the general case to this one. If $u(x)$ is a differentiable function.

$$\frac{d}{dx}\left(u(x)\,y(x)\right) = u(x)\frac{dy}{dx} + y(x)\frac{du}{dx}$$

$$\frac{1}{u(x)}\frac{d}{dx}\left(u(x)\,y(x)\right) = \frac{dy}{dx} + \frac{1}{u}\frac{du}{dx}\,y.$$

If $\frac{1}{u}\frac{du}{dx} = p(x)$; $\frac{du}{u} = p(x)\,dx$

$$u = e^{\int p(x)\,dx}.$$

then:

$$\frac{1}{u(x)}\frac{d}{dx}\left(u(x)\,y(x)\right) = \frac{dy}{dx} + p(x)y = f(x)$$

$$\frac{d}{dx}\left(u(x)\, y(x)\right) = f(x)\, u(x)$$

$$u(x)\, y(x) = \int f(x)\, u(x)\, dx + c$$

$$\boxed{\begin{gathered} y(x) = \frac{1}{u(x)}\left[\int f(x)\, u(x)\, dx + c\right] \\ u(x) = e^{\int p(x)\, dx} \end{gathered}}$$

<u>Example:</u> Solve $\dfrac{dy}{dx} + 2xy = 2x^3$

$$\frac{1}{u(x)}\frac{d}{dx}\left(u(x)\, y\right) = \frac{dy}{dx} + \frac{1}{u}\frac{du}{dx}\, y = 2x^3$$

$$\frac{1}{u}\frac{du}{dx} = 2x \; ; \quad \frac{du}{u} = 2x\, dx$$

$$\ln u = x^2 \; ; \quad \boxed{u = e^{x^2}}$$

$$\frac{d}{dx}\left(e^{x^2} y\right) = 2x^3 e^{x^2}.$$

$$e^{x^2} y = \int 2x^3 e^{x^2}\, dx + c.$$

(3)

$$\int 2x^3 e^{x^2} dx = \int 2x \cdot x^2 e^{x^2} dx$$

$$x^2 = w, \quad dw = 2x\, dx$$

$$= \int w e^w dw = we^w - e^w.$$

$$= x^2 e^{x^2} - e^{x^2}.$$

$$y = x^2 - 1 + c e^{-x^2}.$$

$$\frac{dy}{dx} = +2x - 2x\, c\, e^{-x^2}.$$

$$\frac{dy}{dx} + 2xy = +\cancel{2x} - \cancel{2xc\, e^{-x^2}} + 2x^3 - \cancel{2x}$$

$$+ \cancel{2xc\, e^{-x^2}}$$

$$= 2x^3.$$

$$\cos x \frac{dy}{dx} - \sin x \, y = \sin 2x.$$

divide by $\cos x$: $\quad \frac{dy}{dx} - \tan x \, y = \frac{\sin 2x}{\cos x}$

$$\sin 2x = 2 \sin x \cos x$$

$$\frac{dy}{dx} - \tan x \, y = 2 \sin x.$$

$$\frac{1}{u} \frac{d}{dx}(u y) = 2 \sin x.$$

$$\frac{1}{u} \frac{du}{dx} = - \tan x$$

$$\frac{du}{u} = - \tan x \, dx = - \frac{\sin x}{\cos x} dx$$

$$\ln |u| = \ln |\cos x|$$

$$u = \cos x$$

$$\frac{1}{\cos x} \frac{d}{dx}(\cos x \, y) = 2 \sin x$$

$$\frac{d}{dx}(\cos x \, y) = 2 \sin x \cos x = \sin 2x$$

$$\cos x \, y = \frac{1}{2} \cos 2x + c$$

$$\boxed{y = \frac{1}{2} \frac{\cos 2x}{\cos x} + \frac{c}{\cos x}}$$

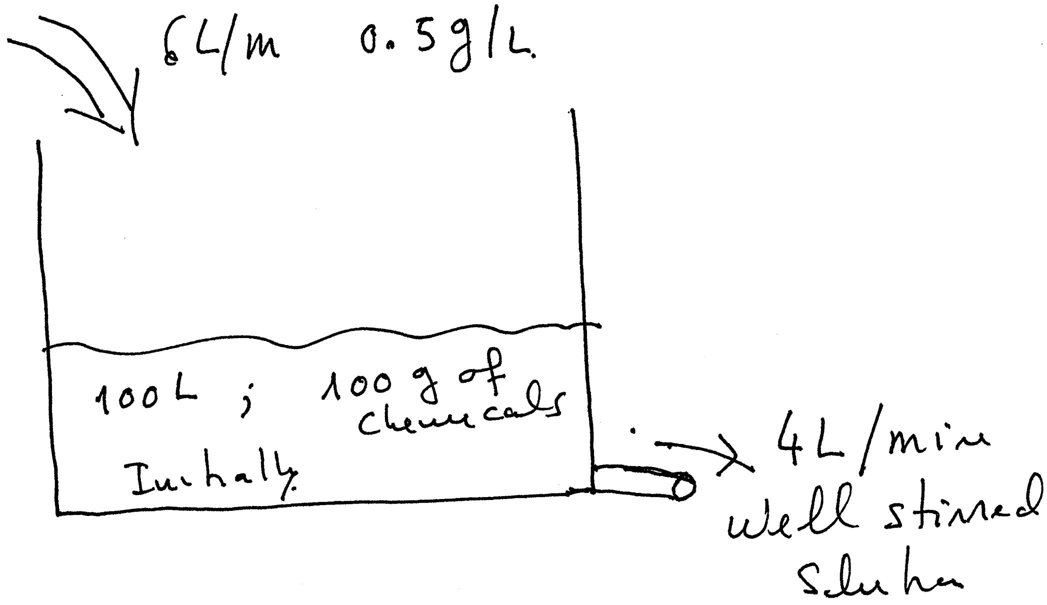
Applications:

Mixing Problems:

Ex: A tank whose Vol is 200L is initially half full of a solution that contains 100 g of chemical.

A sol containing 0.5g/L of the same chemical flows into the tank at a rate of 6 L/min and the well-stirred mixture flows out at a rate of 4 L/min.

Determine the concentration of the chemical in the tank just before the solution overflows.



$V(t)$ = Volume of liquid in tank.

$$\frac{dV}{dt} = \text{In} - \text{Out} = 2 \text{ L/min} \quad ; \; V(0) = 100$$

$$\boxed{V(t) = 100 + 2t}$$

$A(t)$ = Amount of chemical in the tank :

$$\frac{d}{dt} A(t) = A_{In}(t) - A_{out}(t)$$

$$A_{In}(t) = 3 \text{ g/min}$$

$$A_{out} = \frac{A(t)}{V(t)} \cdot \text{Rate of mixture going out}$$

$$\frac{dA}{dt} = 3 - \frac{4\,A(t)}{100 + 2t}.$$

$$\frac{dA}{dt} + \frac{2\,A(t)}{\cancel{100}50+t} = 3$$

$$\frac{1}{u}\frac{du}{dt} = \frac{2}{50+t} \quad ; \quad \ln u = 2 \ln 50+t$$

$$u = (50+t)^2$$

$$\frac{1}{(50+t)^2}\frac{d}{dt}\left((50+t)^2 A \right) = 3$$

$$\frac{d}{dt}\left((50+t)^2 A \right) = 3(50+t)^2$$

$$(50+t)^2 A = (50+t)^3 + C.$$

$$A(t) = 50+t + \frac{C}{(50+t)^2}.$$

$$A(0) = 100 = 50 + \frac{C}{(50)^2}$$

$$C = (50)^3$$

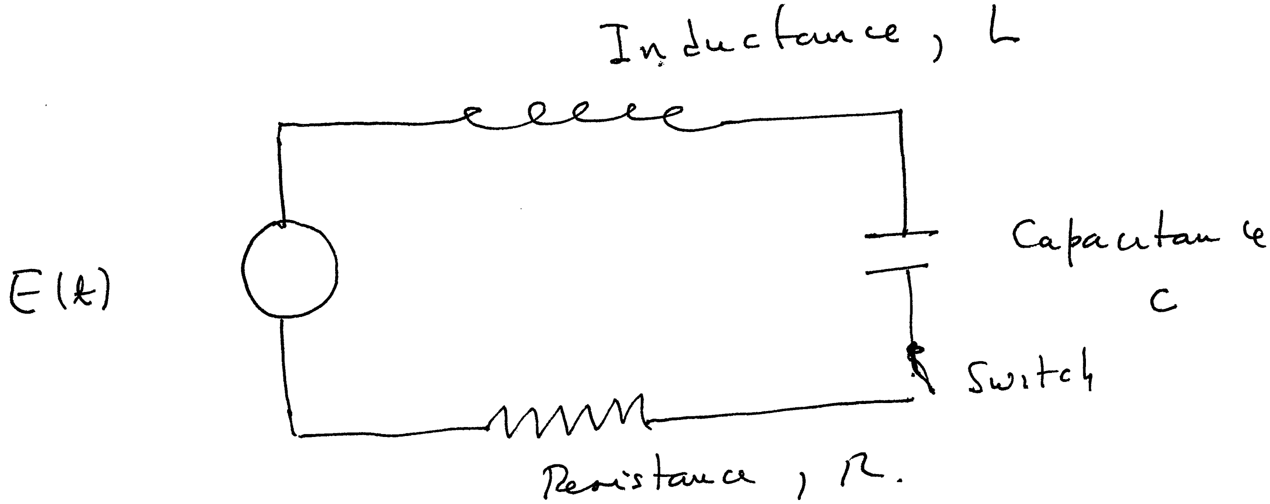
$$A(t) = 50+t + \frac{2500}{(50+t)^2}.$$

tank overflows when $t = 50$

$$A(50) = 100 + \frac{2500}{10000} = 104 \text{ g}.$$

$$C(50) = \frac{104}{200} = 0.52$$

RLC - Circuits



Kirchoff's Law: The sum of voltage drops around a closed circuit is zero

$$\begin{cases} L\dfrac{dI}{dt} + RI + \dfrac{1}{C}q = E(t) \\[2ex] \dfrac{dq}{dt} = I. \end{cases}$$

$I(t)$ = Current;

RL - Circuit.

$$\boxed{\frac{dI}{dt} + \frac{R}{L} \cdot I = \frac{1}{L} E(t)}$$

RC - Circuit: $L = 0$.

$$i + \frac{1}{RC} q = E/R$$

$$\frac{dq}{dt} = i(t)$$

$$\boxed{\frac{dq}{dt} + \frac{1}{RC} q = E/R.}$$