Lesson 4 Section 1.8 Change of Variables

Solve: $\dfrac{dy}{dx} = \dfrac{3y}{3x-2y}$.

Idea: $\dfrac{3y}{3x-2y} = \dfrac{3y}{x(3-2y/x)} = \dfrac{3\,y/x}{3-2y/x}$

Set $V = y/x$; $y = xV$.

$$\frac{dy}{dx} = V + x\frac{dV}{dx}.$$

$$V + x\frac{dV}{dx} = \frac{3xV}{3x-2xV} = \frac{3V}{3-2V}.$$

$$x\frac{dV}{dx} = \frac{3V}{3-2V} - V = \frac{3V-3V+2V^2}{3-2V}.$$

$$x\frac{dV}{dx} = \frac{2V^2}{3-2V}$$

$$\frac{3-2V}{2V^2}\,dV = \frac{dx}{x}.$$

$$\left(\frac{3}{2V^2} - \frac{1}{V}\right) dV = \frac{dx}{x}$$

$$-\frac{3}{2V} - \ln|V| = \ln|x| + c.$$

$$-\ln\left(|V|\, e^{3/2V}\right) = \ln|x|\, e^{c}$$

$$\left(|V|\, e^{3/V}\right)^{-1} = |x|\, e^{c}$$

$$|V|\, e^{3/V} = \frac{K}{x}$$

$$\boxed{V e^{3/V} = K/x}$$

$$x \frac{dy}{dx} - y = \sqrt{9x^2 + y^2} \quad ; \quad x > 0$$

divide by x : $\quad \frac{dy}{dx} - y/x = \sqrt{9 + (y/x)^2}.$

$$y/x = V \,; \quad y = xV \quad \frac{dy}{dx} = V + x\frac{dV}{dx}.$$

$$v + x\frac{dv}{dx} - v = \sqrt{9+v^2} \quad \text{and so}$$

$$x\frac{dv}{dx} = \sqrt{9+v^2}\,; \qquad \frac{dv}{\sqrt{9+v^2}} = \frac{dx}{x}$$

$$\int \frac{dv}{\sqrt{9+v^2}} = \int \frac{3\,dt}{\sqrt{9+9t^2}} = \int \frac{dt}{\sqrt{1+t^2}}\,; \quad t = \tan\theta$$

$$v = 3t \qquad\qquad dt = \sec^2\theta\, d\theta$$

$$\sqrt{1+t^2} = \sec\theta$$

$$= \int \frac{\sec^2\theta}{\sec\theta}\,d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta|$$

$$= \ln\left|t + \sqrt{1+t^2}\right| = \ln\left|v/3 + \sqrt{1+\frac{v^2}{9}}\right|$$

Hence: $\ln\left|v/3 + \sqrt{1+\frac{v^2}{9}}\right| = \ln|x| + c.$

$$\boxed{v/3 + \sqrt{1+\frac{v^2}{9}} = Kx}$$

$$\sqrt{9+v^2} = Kx - v\,; \quad K \text{ constant}$$

$$9+v^2 = (Kx - v)^2; \quad 9 + (y/x)^2 = (Kx - y/x)^2.$$

$$9x^2 + y^2 = (Kx^2 - y)^2. \quad \boxed{9x^2 = K^2x^4 - 2Kx^2y}$$

Homogeneous functions: $f(x,y)$ is homogeneous of degree m if

$$f(tx, ty) = t^m f(x,y)\,; \quad t > 0.$$

$f(x,y)$ is homogeneous of degree m if

$$f(tx, ty) = t^m f(x,y) \; ; \quad t > 0$$

(tx, ty) in the domain of fcn.

Ex. $f(x,y) = \dfrac{\sqrt{9x^2 + y^2}}{x}$; $f(tx,ty) = f(x,y)$

f is homogeneous of degree zero.

If f is homogeneous of degree m then

$$f(x,y) = f(x, x\, y/x) = x^m f(1, y/x)$$

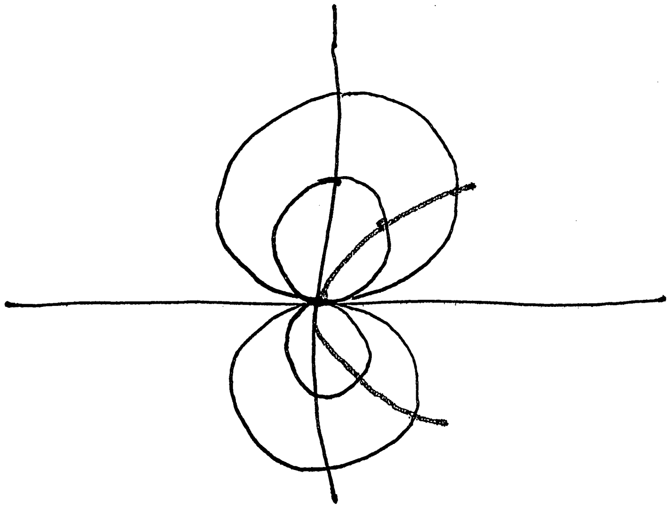
one can think of f as a function of x and y/x

Ex: Determine the orthogonal trajectories to

$$x^2 + y^2 = 2cy$$

$$x^2 + y^2 - 2cy = 0$$

$$x^2 + (y-c)^2 = c^2$$



Tangent to the curve $x^2 + y^2 = 2cy$

$$2x + 2y\frac{dy}{dx} = 2c\frac{dy}{dx}$$

$$(c - y)\frac{dy}{dx} = x$$

$$\frac{dy}{dx} = \frac{x}{c-y} \; ; \qquad c = \frac{x^2+y^2}{2y}$$

$$\frac{x}{c-y} = \frac{x}{\frac{x^2+y^2}{2y} - y} = \frac{2xy}{x^2-y^2}$$

$$\frac{dy}{dx} = \frac{2xy}{x^2-y^2} \; ; \qquad \cancel{y = vx} \qquad \cancel{\frac{dy}{dx} = v + x\frac{dv}{dx}}$$

orthogonal to the curve

$$\frac{dy}{dx} = - \frac{x^2-y^2}{2xy} \quad ; \; y = xV.$$

$$\frac{dy}{dx} = V + x\frac{dV}{dx}$$

$$V + x\frac{dV}{dx} = - \frac{1-V^2}{2V}.$$

$$x\frac{dV}{dx} = \frac{V^2-1}{2V} - V = \frac{V^2-1-2V^2}{2V}$$

$$x\frac{dV}{dx} = - \frac{V^2+1}{2V}.$$

$$\frac{2V\,dV}{1+V^2} = -\frac{dx}{x}$$

$$\ln|1+V^2| = -\ln|x| + K$$

$$V = y/x$$

$$1+V^2 = e^K|x|^{-1}$$

$$1 + y^2/x^2 = C/x \; ; \quad \boxed{x^2+y^2 = Cx}$$

Bernoulli's Equation :

$$\frac{dy}{dx} + p(x)\, y = q(x)\, y^n.$$

Step 1 : divide equation by y^n , $n \neq 1$.

$$\frac{1}{y^n}\frac{dy}{dx} + p(x)\, y^{1-n} = q(x).$$

$$u = y^{1-n} \qquad \frac{du}{dx} = (1-n)\, y^{-n}\frac{dy}{dx}.$$

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{du}{dx}.$$

$$\boxed{\frac{1}{1-n}\frac{du}{dx} + p(x)\, u = q(x)}$$

Solve:

$$\frac{dy}{dx} + \frac{1}{2}(\tan x)\, y = 2 y^3 \sin x.$$

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{2}(\tan x)\, y^{-2} = 2 \sin x$$

$$v = y^{-2} \qquad \frac{dv}{dx} = -2 y^{-3} \frac{dy}{dx}.$$

$$-\frac{1}{2}\frac{dv}{dx} + \frac{1}{2}(\tan x)\, v = 2 \sin x.$$

$$\frac{dv}{dx} - (\tan x)\, v = -4 \sin x$$

$$\cancel{\frac{d}{dx}\left(\log \cos x\right)}$$

$$\frac{1}{\cos x}\frac{d}{dx}\left(\cos x\, v\right) = -4 \sin x$$

$$\frac{d}{dx}\left(\cos x\, v\right) = -4 \sin x \cos x = -2 \sin 2x$$

$$\cos x\, v = \cos 2x + c$$

$$v = \frac{\cos 2x}{\cos x} + c.$$

9.

$$\frac{1}{y^2} = \frac{\cos 2x}{\cos x} + C.$$

$$\boxed{y = \left[C + \frac{\cos 2x}{\cos x} \right]^{-1/2}.}$$