Lesson 5 Section 1.9 Exact Equations

$$M(x,y)\,dx + N(x,y)\,dy$$

is an exact form if there exists a function $F(x,y)$ such that

$$M(x,y)\,dx + N(x,y)\,dy = dF = \frac{\partial F}{\partial x}(x,y)\,dx + \frac{\partial F}{\partial y}(x,y)\,dy$$

$$\boxed{\frac{\partial F}{\partial x} = M \; ; \; \frac{\partial F}{\partial y} = N}$$

Test for exactness. Let $M(x,y)$; $N(x,y)$ and $\nabla M, \nabla N$ be continuous in a region R of the plane which has no holes.

Then

$$M\,dx + N\,dy$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example - Determine whether this form is exact and find F

(1) $(y + 3x^2)\, dx + x\, dy$

Sol : $M = y + 3x^2$; $N = x$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 1.$$

Domain of M, N is R^2 which is simply connected.

Find F such that

$$\frac{\partial F}{\partial x} = y + 3x^2 ; \quad \frac{\partial F}{\partial y} = x.$$

$$\frac{\partial F}{\partial x} = y + 3x^2 ; \quad F = xy + x^3 + \psi(y)$$

$$\frac{\partial F}{\partial y} = x + \psi' = x ; \quad \psi' = 0,$$

$$\psi = c.$$

$$\boxed{F(x,y) = xy + x^3 + c}$$

Solve the differential equation

5). $(y^2 + \cos x)\,dx + (2xy + \sin y)\,dy = 0$ (3)

$y(0) = \pi$.

$M = y^2 + \cos x$; $N = 2xy + \sin y$

$\frac{\partial M}{\partial y} = 2y$; $\frac{\partial N}{\partial x} = 2y$

Domain: $\mathbb{R}^2$ simply connected.

$\frac{\partial F}{\partial x} = y^2 + \cos x$; $F(x,y) = xy^2 + \sin x + \psi(y)$

$\frac{\partial F}{\partial y} = 2xy + \psi'(y) = 2xy + \sin y$

$\psi'(y) = \sin y$; $\psi = -\cos y + C$.

So $\boxed{F(x,y) = xy^2 + \sin x - \cos y}$

$\cancel{F(0,\pi) = -\cos\pi + C}$

Solution: $xy^2 + \sin x - \cos y = C$

$x = 0,\ y = \pi$; $-\cos\pi = C$, $C = 1$.

$$\boxed{xy^2 + \sin x - \cos y = 1}$$

Integrating Factor.

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\partial M/\partial y \neq \frac{\partial N}{\partial x}$$

Is it possible to

find a function $I(x,y)$ such that

$$I(x,y)\,M(x,y)\,dx + N(x,y)\,I(x,y)\,dy$$

is exact? In other words

$$\frac{\partial}{\partial y}(I\,M) = \frac{\partial}{\partial x}(I\,N)$$

$$I\frac{\partial M}{\partial y} + M\frac{\partial I}{\partial y} = I\frac{\partial N}{\partial x} + N\frac{\partial I}{\partial x}$$

$$\boxed{I\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial I}{\partial x} - M\frac{\partial I}{\partial y}}$$

This can be very complicated to solve.

Simpler question: Can we find $I = I(x)$?

$$I(x)\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial I}{\partial x}$$

In this case

$$\frac{1}{I}\frac{\partial I}{\partial x} = \frac{\partial M/\partial y - \frac{\partial N}{\partial x}}{N} = \varphi(x) \text{ only}$$

Similarly

$$\frac{1}{I}\frac{\partial I}{\partial y} = \frac{\partial N/\partial x - \frac{\partial M}{\partial y}}{M} = \psi(y)!$$

Example :

$$(3xy - 2/y)\, dx + x(x + y^{-2})\, dy = 0$$

$$y(1) = 3$$

$$M = 3xy - 2/y \quad ; \quad N = x^2 + xy^{-2}$$

$$\partial M/\partial y = 3x + 2/y^2 \quad ; \quad \partial N/\partial x = 2x + y^{-2}$$

$$\partial M/\partial y - \frac{\partial N}{\partial x} = x + y^{-2}$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{x + y^{-2}}{x^2 + xy^{-2}} = 1/x$$

$$\frac{1}{I}\frac{dI}{dx} = 1/x \quad ; \quad \ln|I| = \ln|x| + c$$

$$I = x.$$

Multiply equation by x.

$$x\left(3xy - 2/y\right)dx + x^2(x + y^{-2})\,dy = 0$$

$$M = 3x^2y - 2xy^{-1} \quad ; \quad N = x^3 + x^2y^{-2}$$

$$\partial M/\partial y = 3x^2 + 2xy^{-2} \quad ; \quad \frac{\partial N}{\partial x} = 3x^2 + 2xy^{-2}$$

Find F such that

$$\frac{\partial F}{\partial x} = 3x^2y - 2xy^{-1} \quad ; \quad \frac{\partial F}{\partial y} = x^3 + x^2y^{-2}$$

$$F = x^3y - x^2y^{-1} + \varphi(y)$$

$$\partial F/\partial y = x^3 + x^2y^{-2} + \varphi'(y) = x^3 + x^2y^{-2}$$

$$\varphi' = 0$$

Hence the solution satisfies.

$$F(x,y) = x^3y - x^2y^{-1} = C.$$

$$y(1) = 3 \quad ; \quad F(1,3) = 3 - \frac{1}{3} = 8/3$$

$$\boxed{x^3y - x^2y^{-1} = 8/3}$$

Solve:

$$\frac{dy}{dx} + p(x)y = q(x)$$

by this method.

$$\frac{dy}{dx} + p(x)y - q(x) = 0$$

$$dy + \left(p(x)y - q(x)\right)dx = 0$$

$$M(x,y) = p(x)y - q(x) \; ; \quad N = 1.$$

$$\frac{\partial M}{\partial y} = p(x) \; ; \quad \frac{\partial N}{\partial x} = 0. \qquad \underline{\text{Not exact}}$$

<u>Integrating factor:</u> $\frac{1}{I}\frac{dI}{dx} = \frac{\partial M/\partial y - \partial N/\partial x}{N} = p(x)$

$$\frac{1}{I} \cdot \frac{dI}{dx}(x) = p \; ; \quad \frac{d}{dx}\ln I = p(x)$$

$$\boxed{I = e^{\int p(x)\,dx}}$$

$$I\,dy + I\left(p(x)y - q(x)\right)dx = 0 \quad \text{is exact}$$

$$\frac{\partial}{\partial y}F(x,y) = I. \; ; \quad F(x,y) = yI + \psi(x)$$

$$\frac{\partial F}{\partial x} = \cancel{y \frac{\partial I}{\partial x}} + \frac{\partial \psi}{\partial x} = \left(\cancel{p(x) y} - q(x)\right) I.$$

$$\frac{dI}{dx} = p\, I$$

$$\frac{\partial \psi}{\partial x} = -\, q(x)\, I.$$

$$\boxed{I\, y - \int q(x)\, I(x)\, dx = c.}$$