Lesson 7: Sections 1.1 and 1.2

Section 1.1: Matrices: Definitions and Notation

$$\begin{array}{c} \text{Columns} \\ A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array} \to \text{Rows}$$

A is an $m \times n$ matrix

A = Matrix is a rectangular array of numbers
m rows, n columns

Ex:
$$A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 5 \end{bmatrix}$$
3 rows, 3 columns

$A = 3 \times 3$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & \pi & \sqrt{2} & 3 \end{bmatrix}$$
2×4 matrix

When $m = n$, A is said to be a square ~~rectangular~~ matrix.

Two matrices $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & & a_{mn} \end{bmatrix}$; $B = \begin{bmatrix} b_{11} & \cdots & b_{1l} \\ \vdots & & \\ b_{k1} & \cdots & b_{kl} \end{bmatrix}$

$A = B$ if $m = k$; $n = l$

$a_{ij} = b_{ij}$

A matrix $A = [a_1 \ a_2 \ \cdots \ a_n]$ = Row n-vector

$$A = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}$$ Column ~~Rows~~ m- vector.

$$A = \begin{bmatrix} 1 & 0 & \sqrt{2} \\ 5 & 8 & \sqrt{3} \\ 3 & 1 & 9 \end{bmatrix}$$ is formed by 3 Row 3-vectors
3 Column 3-vectors

$$V_1 = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}; \quad V_2 = \begin{bmatrix} 0 \\ 8 \\ 1 \end{bmatrix}, \quad V_3 = \begin{bmatrix} \sqrt{2} \\ \sqrt{3} \\ 9 \end{bmatrix}$$

$$R_1 = [1, \ 0 \ \ \sqrt{2}] \ ; \quad R_2 = [5 \ \ 8 \ \ \sqrt{3}]$$

$$R_3 = [3 \ \ 1 \ \ 9].$$

<u>THE TRANSPOSE of a matrix</u>: A^T is obtained from A by interchanging Rows and Columns

$$\underset{2\times 3}{A} = \begin{bmatrix} 1 & 3 & 4 \\ 8 & 1 & 2 \end{bmatrix}$$

$$\underset{3\times 2.}{A^T} = \begin{bmatrix} 1 & 8 \\ 3 & 1 \\ 4 & 2 \end{bmatrix}$$

Square Matrices

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

\# of rows = \# of columns.

$$Tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

Diagonal $\quad a_{11} \cdots a_{nn}$

Special square matrices:

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & 0 \\ 0 & 0 & \cdots & & a_{nn} \end{bmatrix}$$

Diagonal matrix,

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

upper triangular.

$Tr(A) = 6$

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 6 & 0 \\ 0 & 1 & 4 & 8 \end{bmatrix}$$

Lower triangular

$Tr(A) = 20$

<u>Symmetric Matrices:</u> A square matrix is symmetric if $A^T = A$

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 2 & 4 & 9 \\ 1 & 4 & 5 & \sqrt{2} \\ 3 & 9 & \sqrt{2} & 7 \end{bmatrix}$$

Row 1 = Column 1

Row 2 = column 2

<u>Matrix Function:</u> $M = \left(a_{jk}(t) \right)_{m \times n}$.

<u>Operations with Matrices:</u>

(1) <u>Addition:</u> If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices,

$$A + B = [a_{ij} + b_{ij}] \; ; \; A - B = [a_{ij} - b_{ij}]$$

<u>Ex:</u>

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 5 & 7 \\ 1 & 3 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 7 & 10 \\ 5 & 11 & 5+\sqrt{2} \end{bmatrix}$$

A and B must be of the same size.

Scalar multiplication $s[a_{ij}] = [s\,a_{ij}]$

$$\sqrt{2}\begin{bmatrix} 5 & 3 & 4 \\ 1 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 5\sqrt{2} & 3\sqrt{2} & 4\sqrt{2} \\ \sqrt{2} & 0 & 7\sqrt{2} \end{bmatrix}$$

Multiplication:

(I) Row n-vector times Column n-vector

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Eg: $\begin{bmatrix} 1 & 5 & 3 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 1 \\ 0 \end{bmatrix} = 8 + 5 + 0 = 13.$

(II) ($m \times n$ matrix M) x Column n-vector V

$$\begin{array}{c} R_1 \to \\ R_2 \\ \vdots \\ R_m \end{array} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} R_1 \cdot V \\ R_2 \cdot V \\ \vdots \\ R_m \cdot V \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & 1 & 8 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+2+6 \\ 2-3+15 \\ 2+2-8+3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 14 \\ -1 \end{bmatrix}$$

(III) $\cancel{A} A = m \times n$ matrix $B = n \times k$

$$A \cdot B = m \times k$$

$$\begin{array}{c} R_1 \to \\ R_2 \to \\ \\ R_m \end{array} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix} \cdot \begin{array}{ccc} C_1 & C_2 & C_m \\ \downarrow & \downarrow & \downarrow \end{array} \begin{pmatrix} b_{11} & b_{12} \cdots & b_{1k} \\ b_{21} & b_{22} \cdots & b_{2k} \\ b_{n1} & b_{n2} \cdots & b_{nK} \end{pmatrix}$$

$$= \begin{bmatrix} A\,C_1 & A\,C_2 & \cdots & A\,C_K \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 & 2 \\ 1 & 8 & 7 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & 8 \\ 4 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} =$$

$$2 \times 3 \qquad\qquad 3 \times 3$$

$$\begin{bmatrix} A c_1 & A c_2 & A c_3 \end{bmatrix} = \begin{bmatrix} 28 & 6 & 41 \\ 48 & 8 & 37 \end{bmatrix}$$

$$A c_1 = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 8 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 12+12+4 \\ 3+32+14 \end{bmatrix}$$

$$= \begin{bmatrix} 28 \\ 48 \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$