The Cotangent Bundle — Lesson 10

$$T^*X = (TX)^* = \text{Phase space.}$$

Sections of T^*X are called 1-forms.

<u>Examples</u>: $\varphi \in C^\infty(X)$, $d\varphi$ is the 1-form defined by

$$d\varphi(V) = V\varphi$$

The canonical 1-form

$$\begin{array}{c} T^*X \\ \downarrow \pi \\ X \end{array} \qquad d_p\pi : T_p(T^*X) \to T_m(X)$$

$$\cancel{d_p\pi} : T_p(T^*X) \xrightarrow{d_p\pi} T_m(X) \xrightarrow{p} \mathbb{R}$$

$$p \circ d_p\pi : T_p(T^*X) \longrightarrow \mathbb{R}.$$

is a 1-form on T^*X

<u>Local Coordinates</u>: $TX \quad (x_1, x_2, \ldots, x_n, v_1, v_2, \ldots, v_n)$

$$V = \sum v_i \, \partial/\partial x_i$$

T^*X: $(x_1, x_2, \ldots, x_n, \xi_1, \xi_2, \ldots, \xi_n)$

$$\gamma = \sum_{j=1}^{n} \xi_j \, dx_j \in T^*X$$

$$\gamma(v) = \sum \xi_j v_j$$

think of $\gamma \in T^*(T^*X)$, think of

$(\xi_j\, x_j)$ as functions on $\mathbb{R}^n \times \mathbb{R}^n$.

$$\pi(x_1, x_2, \ldots x_n, \xi_1, \xi_2, \ldots, \xi_n) = (x_1, x_2, \ldots, x_n).$$

$$d_\gamma \pi . \left(\sum a_j \partial_{x_j} + b_j \partial_{\xi_j} \right) = \sum a_j \partial_{x_j}$$

$$\gamma \circ d_\gamma \pi \, V = \sum \xi_j a_j$$

$$= \left(\sum \xi_j \, dx_j \right) V$$

This identifies $\gamma = \sum_{j=1}^{n} \xi_j \, dx_j$ with the one-form on $T^*(T^*X)$.

Exterior Differentiation:

Let X be a C^∞ manifold. We define $\Lambda^k T^* X$ to be the bundle over $T^* X$ for which the fibers are alternating multilinear form on $T_x X$.

$$\xi_1, \dots, \xi_k \in T_x^* X$$

$$v_1, \dots, v_k \in T_x X$$

$$\xi_1 \wedge \xi_2 \wedge \dots \wedge \xi_k : T_x X \times T_x X \dots \times T_x X \longrightarrow \mathbb{R}$$

$$(v_1, \dots, v_k) \longmapsto \det\left[\xi_i(v_j)\right]$$

In local coordinates. $x_1, x_2, x_3, \dots, x_n$

$$\omega \in \Lambda^k T^* X \iff$$

$$\omega = \sum_{j_l \in \{1,2,\dots,k\}} f_{j_1 \dots j_k} \, dx_{j_1} \wedge dx_{j_2} \wedge \dots \wedge dx_{j_k}$$

Exterior differentiation: Is a differential operator:

$$d : \Lambda^k T^* X \longrightarrow \Lambda^{k+1} T^* X$$

$$d (f_0 \, df_1 \wedge df_2 \wedge \ldots \wedge df_k)$$

$$= df_0 \wedge df_1 \wedge df_2 \wedge \ldots \wedge df_k .$$

Examples: $k = 1$: $\Lambda^1 T^* X = d$ functions

$$\alpha \in \Lambda^1 T^* X$$

$$\alpha = \sum_{j=1}^{n} f_j \, dx_j$$

$$d\alpha = \sum_{j,k=1}^{n} \frac{\partial f_j}{\partial x_k} dx_k \wedge dx_j$$

$k = 2$: $\quad \alpha = \sum_{j,k=1}^{n} f_{jk} \, dx_j \wedge dx_k$

$$d\alpha = \sum_{j,k,l=1}^{m} \frac{\partial f_{jk}}{\partial x_l} dx_l \wedge dx_j \wedge dx_k .$$

$d^2\alpha = 0.$

$$\alpha = \sum_{j=1}^{n} f_j \, dx_j$$

$$d\alpha = \sum_{j=1,k}^{n} \frac{\partial f_j}{\partial x_k} \, dx_k \wedge dx_j$$

$$d^2\alpha = \sum_{j,k,l} \frac{\partial^2 f_j}{\partial x_k \partial x_l} \, dx_l \wedge dx_k \wedge dx_j$$

$$= \sum_j \left(\sum_{k,l} \frac{\partial^2 f_j}{\partial x_k \partial x_l} \, dx_l \wedge dx_k \right) \wedge dx_j$$

$$= \sum_j \left(\sum_{k>l} \underbrace{\left(\frac{\partial^2 f_j}{\partial x_k \partial x_l} - \frac{\partial^2 f_j}{\partial x_l \partial x_k} \right)}_{=0} dx_l \wedge dx_k \right) \wedge dx_j$$

<u>Poincaré's Lemma:</u> If $f \in \Lambda^k T^*X$ and $df = 0$, then for every $p \in X$, there exists a neighborhood U of p ~~such that~~ and $\phi \in \Lambda^{k-1} T^*X$ such that $f = d\phi$.

Example: $k=1$ $\qquad \alpha = \sum_{j=1}^{n} f_j \, dx_j$

$$d\alpha = \sum_{j,k=1}^{n} \frac{\partial f_j}{\partial x_k} \, dx_k \wedge dx_j$$

$$= \sum_{j<k} \left(\frac{\partial f_j}{\partial x_k} - \frac{\partial f_k}{\partial x_j} \right) dx_k \wedge dx_j = 0$$

$$\frac{\partial f_j}{\partial x_k} = \frac{\partial f_k}{\partial x_j} \qquad \nabla \times f = 0$$

The Symplectic Form: X a C^∞ manifold

$T^* X$ its cotangent bundle.

$\alpha \in T^*(T^* X)$ the canonical 1-form. In local coordinates

$$\alpha = \sum_{j=1}^{n} \xi_j \, dx_j$$

$$\omega = d\alpha = \sum_{j=1}^{n} d\xi_j \wedge dx_j .$$

ω = Canonical 2-form, Symplectic form.

ω is anti-symmetric and non-degenerate.

$$\omega(u, v) = -\omega(v, u)$$

$$\omega(u,v) = \sum d\xi_j(u)\, dx_j(v) - d\xi_j(v)\, dx_j(u)$$

Suppose for fixed u

$\omega(u, v) = 0$ for all v.

In particular, true for $v = \partial_{x_\ell} \Rightarrow d\xi_\ell(u) = 0$
$\ell = 1, 2, \ldots, n.$

$v = \partial_{\xi_\ell}$ $\quad dx_\ell(u) = 0 \quad \Rightarrow u = 0.$

<u>Symplectic Linear Algebra</u> Let S be a vector space. Suppose S is equipped with a non-degenerate anti-symmetric bilinear form σ. $(S, \sigma) \equiv$ symplectic vector space.

Example: $T^*\mathbb{R}^n \equiv \mathbb{R}^{2n} = \{ (x,\xi) : x, \xi \in \mathbb{R}^n \}$

$$\sigma_0 \big((x,\xi); (x',\xi') \big) = \langle x', \xi \rangle - \langle x, \xi' \rangle .$$

$$x = \sum_{j=1}^{n} x_j e_j \quad ; \quad \xi = \sum_{j=1}^{n} \xi_j \varepsilon_j$$

$$\sigma_0 (e_j , e_k) = 0 \quad , \quad \sigma_0 (\varepsilon_j, \varepsilon_k) = 0$$

$$\sigma_0 (\varepsilon_j, e_k) = -\sigma_0 (e_k, \varepsilon_j) = \delta_{jk}$$

Proposition: If (S,σ) is a symplectic vector space. Then S has even dimension $2n$ and there exists a linear map

$$T : (S,\sigma) \longrightarrow (T^* \mathbb{R}^n, \sigma_0)$$

which is invertible and such that $T^* \sigma_0 = \sigma$ in the sense that

$$\sigma(v,w) = \sigma_0 (Tv, Tw)$$

Proof. Pick $\varepsilon_1 \in S$, Since σ is non-degenerate (5)

we can choose $e_1 \in S$ such that

$$\sigma(\varepsilon_1, e_1) = 1.$$

Let $S_1 = \mathrm{Span}\{e_1, \varepsilon_1\}$. $\dim S_1 = 2$.

Since $\sigma(e_1, e_1) = \sigma(\varepsilon_1, \varepsilon_1) = 0$.

Let $\tilde{S} = S_1^{\perp} = \{\gamma \in S : \sigma(\gamma, S_1) = 0\}$

$\tilde{S}$ is supplementary to S_1: If $\gamma \in \tilde{S} \cap S_1$,

$\gamma = a e_1 + b \varepsilon_1$ $\qquad \gamma = 0$

$\sigma(\gamma, \varepsilon_1) = -a = 0$, $\sigma(\gamma, e_1) = b = 0$

<u>$\tilde{S}$ is a symplectic space</u> Check that $\sigma|_{\tilde{S}}$ is non-degenerate.

If $v \in \tilde{S}$, $\sigma(v, w) = 0 \;\; \forall\, w \in \tilde{S}$

then $\sigma(v, w) = 0 \;\; \forall\, w \in S \Rightarrow v = 0$

~~Also~~ Suppose we have proved the proposition for $< \dim S$, choose a basis $\{e_2, \varepsilon_2, e_3, \varepsilon_3, \dots, \varepsilon_n, \varepsilon_n\}$