The Lie Derivative of Differential Forms. Lesson 12

X a C^∞ manifold

$\Lambda^k X = \Lambda^k T^* X$ = Vector bundle whose

Sections are k-forms. In local coordinates

$\omega \in \Lambda^k X \qquad \omega = \sum_{j_k=1}^{k} f_{j_1 j_2 \cdots j_k} \, dx_{j_1} \wedge dx_{j_2} \cdots dx_{j_k}.$

Exterior Differentiation:

$$d : \Lambda^k M \longrightarrow \Lambda^{k+1} M$$

$$d\omega = \sum df_J \wedge dx_{j_1} \wedge dx_{j_2} \cdots \wedge dx_{j_k}$$

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n .$$

~~Proposition~~ <u>Pull Back:</u> $\Phi : \cancel{M} \longrightarrow Y$

a diffeomorphism:

$\omega \in \Lambda^k Y$

$\Phi^* \omega \in \Lambda^k X$

$$\Phi^* \omega(x)(v_1, v_2, \dots, v_k)(x) = \omega(y)(D\Phi(x)v_1, \dots, D\Phi(x)v_k)$$ ②

Proposition:

(1) $\Phi^*(\alpha \wedge \beta) = \Phi^*\alpha \wedge \Phi^*\beta \qquad \alpha \in \Lambda^k Y,\ \beta \in \Lambda^\ell Y$

(2) $\Phi^* d\alpha = d\,\Phi^*\alpha \qquad \forall\, \alpha \in \Lambda^k Y$

Proof: (1) ~~is just~~ follows from the definition.

Let $v \in C^\infty(X, T^*X)$ and let

$\Phi_v^t : X \longrightarrow X$ be the flow of V

$$\mathcal{L}_v\, \omega = \frac{d}{dt} \Phi_t^* \omega \Big|_{t=0} \in \Lambda^k X.$$

Contraction: ~~for~~ $\omega \in \Lambda^k X, \quad v \in C^\infty(X, TX)$

$\omega | v$

$\{\omega | v\} \in \Lambda^{k-1} X$

$$(\omega | v)(x)(e_1, \dots, e_{k-1}) = \omega(x)(v, e_1, e_2, \dots, e_{k-1}).$$

<u>Proposition:</u> If $\omega \in \Lambda^k X$ and $v \in C^\infty(X, TX)$

$$\mathcal{L}_v \omega = d(\omega | v) + (d\omega) | v.$$

<u>Proof:</u> <u>Step 1</u>: $d(\mathcal{L}_v \omega) = \mathcal{L}_v d\omega$.

<u>Step 2</u>: If $\Phi : X \longrightarrow Y$ is a diffeomorphism

(1) $$\Phi^*(\mathcal{L}_v \omega) = \mathcal{L}_{\Phi^* v} \Phi^* \omega.$$

(2) $$\Phi^*(\omega | v) = (\Phi^* \omega, \Phi^* v).$$

$\forall\ \omega \in \Lambda^k Y$, $v \in C^\infty(Y, TY)$

<u>Pf:</u> <u>Exercise</u>, follows from the definition of $\Phi^*\omega$ and $\Phi^* v$.

Proof is by induction in k

$k=0$: $\quad \Lambda^0 X = C^\infty(X)$

. $\omega \in C^\infty(X)$

$$\frac{d}{dt}\left(\Phi_v^{t*}\,\omega\right)\Big|_{t=0} = \frac{d}{dt}\,\omega(\gamma(t,x))\Big|_{t=0} = d\omega(v)$$

$$= (d\omega\,|\,v).$$

Suppose this is true for $\Lambda^{k-1}X$. This is a local statement and $\mathcal{L}_V$ is linear. So it is only necessary to prove this for

$$\omega = f\,d\alpha \qquad \alpha \in \Lambda^{k-1}X.$$

$$\mathcal{L}_V(f\,d\alpha) = \frac{d}{dt}(\Phi^t)^*[f\,d\alpha]\Big|_{t=0} =$$

$$= \frac{d}{dt}\Big((f\circ\Phi^t).(\Phi^t)^*[d\alpha]\Big)\Big|_{t=0}$$

$$= df(v)\,d\alpha + f\,\mathcal{L}_V(d\alpha)$$

$=$

But

$$\mathcal{L}_V \, d\alpha = d(\mathcal{L}_V \alpha)$$

By the induction hypothesis

$$\mathcal{L}_V \alpha = d(\alpha | v) + (d\alpha) | v.$$

$$d(\mathcal{L}_V \alpha) = d((d\alpha) | v).$$

So we conclude that

$$\boxed{\mathcal{L}_V (f \, d\alpha) = d f(V) \circ d\alpha + f \, d((d\alpha)|v)}$$

We want to prove that

$$\mathcal{L}_V (f \, d\alpha) = d(f \, d\alpha | v) + d(f \, d\alpha) | v.$$

But

$$d(f \, d\alpha) | v = (df \wedge d\alpha) | v =$$

$$(df \wedge d\alpha | v)(e_1, e_2, \ldots, e_K)$$

$$= (df \wedge d\alpha)(v, e_1, e_2, \ldots, e_K)$$

$$= df(v)\, d\alpha(e_1, \dots, e_k)$$

$$- \sum_j (-1)^j\, d\alpha(v, e_1, \dots, e_{j-1}, e_{j+1}, \dots e_k)\, df(e_j)$$

$$= df(v)\,(d\alpha) - df \wedge d(\alpha | v).$$

So

$$d(f\, d\alpha) | v + d(f\, d\alpha | v) =$$

$$= df(v)\,(d\alpha) - df \wedge d(\alpha | v) + d(f\, d\alpha | v)$$

$$= df(v)\,(d\alpha) + f\, d(d\alpha | v).$$

$$= \mathcal{L}_v(f\, d\alpha).$$