# Symplectic Geometry: Lesson 13



A manifold $S$ equipped with a closed, non-degenerate two form $\sigma$ is called symplectic.

If $(S_1, \sigma_1)$ and $(S_2, \sigma_2)$ are symplectic manifolds, a map

$$\chi : S_1 \longrightarrow S_2$$

which is a diffeomorphism and is such that

$\chi^* \sigma_2 = \sigma_1$ is called a symplectomorphism or a canonical transformation.

Ex: $X$ $C^\infty$ manifold, $S = T^*X$ equipped with the canonical 2 form $\sigma = d\alpha$

$\alpha$ canonical 1-form is a symplectic manifold.

In local coordinates $(x, \xi)$ in $T^*X$

$$\sigma = \sum_{j=1}^{n} d\xi_j \wedge dx_j$$

<u>Definition:</u> Let $(S, \sigma)$ be a symplectic manifold and $v \in C^\infty(S, TS)$ be a vector field on $S$. $v$ is called a symplectic vector field if $\Phi_v^t$ are symplectic transformation.

$$\Phi_v^t : (\mathcal{U}_t, \sigma) \longrightarrow (S, \sigma)$$

$$\mathcal{U} = \{ (t, x) \in \mathbb{R} \times S : t \in I_x \}$$

$$\mathcal{U}_t = \{ x \in S : (t, x) \in \mathcal{U} \}$$

<u>Theorem:</u> The following are equivalent.

(1) $v$ is symplectic

(2) $\mathcal{L}_v \sigma = 0$

(3) $d(\sigma \rfloor v) = 0$

<u>Proof:</u> (1) $\Leftrightarrow$ (2)

$$\mathcal{L}_v \sigma = d(\sigma | v) + (d\sigma) | v$$

but $d\sigma = 0$ so $\mathcal{L}_v \sigma = d(\sigma | v)$

Hence, if $v$ is a symplectic vector field. (3)

$$d(\sigma \lrcorner v) = 0 \quad \Rightarrow \quad \sigma \lrcorner v = df, \quad f \in C^\infty(\mathcal{O})$$

$\mathcal{O} \subset S$ an open set.

<u>Definition:</u> Let $f \in C^\infty(S)$, the Hamiltonian vector field $H_f$ is the unique vector field $v$ on $S$ such that

$$\sigma \lrcorner v = -df$$

<u>Example:</u> $S = T^*X$; $\sigma = \sum_{j=1}^{n} d\xi_j \wedge dx_j$

$$v = \sum_{k=1}^{n} a_k \partial_{x_k} + b_k \partial_{\xi_k}$$

$$(\sigma \lrcorner v) = \sigma(v, \cdot) = \sum b_i\, dx_i - a_i\, d\xi_i$$

$$= df = -\sum \frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial \xi_i} d\xi_i$$

$$b_i = -\frac{\partial f}{\partial x_i}; \quad a_i = \frac{\partial f}{\partial \xi_i}$$

$$H_f = \sum_{k=1}^{n} \frac{\partial f}{\partial \xi_i} \partial/\partial x_i - \frac{\partial f}{\partial x_i} \partial/\partial \xi_i$$

Theorem: Let $(S_1, \sigma_1)$ and $(S_2, \sigma_2)$ be symplectic manifolds and let

$$\Phi : S_1 \longrightarrow S_2$$

be a symplectomorphism. Then if $f \in C^\infty(S_2)$

$$\text{(A)} \qquad \Phi^* H_f = H_{\Phi^* f}$$

Conversely, if (A) holds for $f_j$, $1 \le j \le n$, where $d(\Phi^* f_j)_x$ span $T_x S_1$ $\forall\, x \in S_1$, and $\dim S_1 = \dim S_2$, then $\Phi$ is a symplectomorphism.

Proof: We know that for any $\Phi$ and any $v$ and $\alpha$

$$\Phi^* d\alpha = d\, \Phi^* \alpha.$$

$$\Phi^* (\alpha \lrcorner v) = \Phi^* \alpha \lrcorner \Phi^* v.$$

Hence $\Phi^* \sigma_2 \lrcorner \Phi^* H_f = \Phi^* (\sigma_2 \lrcorner H_f) = -\Phi^* df$

$$= -d\, \Phi^* f.$$

So, if $\Phi^* \sigma_2 = \sigma_1$, then by definition

$$H_{\Phi^* f} = \Phi^* H_f.$$

Reciprocally, if ~~$d$~~ $d(\Phi^* f_i)_x$ span $T_x S_1$, then $D\Phi_x$ is injective and hence.

$H_{f_i}$ span $T_{\Phi(x)} S_2$. and $\Phi^* H_{f_i}(x)$ span $T_x S_1$. and $\Phi^* \sigma_2 - \sigma_1 = 0$ on $T_x S_1$.

<u>Definition</u>: Let $(S, \sigma)$ be a symplectic manifold and $f, g \in C^\infty(S)$, the Poisson bracket $\{f, g\} \in C^\infty(S)$ is defined to be.

$$\{f, g\} = \sigma(H_f, H_g) = H_f\, g.$$

<u>Theorem</u>: If $(S, \sigma)$ is a symplectic manifold $f, g \in C^\infty(S)$,

$$H_{\{f,g\}} = [H_f, H_g]$$

<u>Proof</u>: Let $\Phi^t$ be the $H_f$ flow, then by definition

$$[H_f, H_g] = \frac{d}{dt} \Phi^{t*} H_g \Big|_{t=0} = \frac{d}{dt} H_{\Phi^{t*} g} \Big|_{t=0}$$

$$= H_{H_f g} = H_{\{f,g\}}.$$