Symplectic Geometry: Lesson 14

Theorem: Let (S, σ) be a symplectic manifold of dimension $2n$. Let

$$\omega = \frac{1}{n!}(-1)^{\binom{n}{2}} \sigma \wedge \sigma \wedge \dots \wedge \sigma \quad (n\text{-times})$$

Then there exist local coordinates (x, ξ) valid near each $p \in S$ such that

$$\omega = d\xi_1 \wedge d\xi_2 \dots \wedge d\xi_n \wedge dx_1 \wedge \dots \wedge dx_n.$$

$\omega =$ Canonical volume form

Proof: In canonical coordinates

$$\omega = \underbrace{\left(\sum_{j=1}^{n} d\xi_j \wedge dx_j\right) \wedge \dots \wedge \left(\sum_{j=1}^{n} d\xi_j \wedge dx_j\right)}_{n\text{-times}}$$

$$= n!\,(-1)^{\binom{n}{2}} d\xi_1 \wedge d\xi_2 \dots \wedge d\xi_n \wedge dx_1 \wedge \dots \wedge dx_n.$$

Prop: Let Φ^t be the flow of a symplectic vector field. Then $(\Phi^t)^* \omega = \omega$ for all t.

Lagrangian Submanifolds

Let (E, σ) be a symplectic vector space. If $V \subset E$ is a subspace,

$$V^\sigma = \{ e \in E : \sigma(e, v) = 0 \ \forall v \in V \}.$$

Exercise: A) If $V \subset W \subset E$, then $W^\sigma \subset V^\sigma$

B) $(V^\sigma)^\sigma = V$

(C) $(V \cap W)^\sigma = V^\sigma + W^\sigma$

(D) $(V + W)^\sigma = V^\sigma + W^\sigma$.

(E) $\dim V^\sigma = \dim E - \dim V$.

Definition A subspace $V \subset E$ is

(i) Isotropic, if $V \subset V^\sigma$.

(ii) Involutive if $V^\sigma \subset V$

(iii) Lagrangian if $V^\sigma = V$.

(iv) Symplectic if $V \cap V^\sigma = \{0\}$

<u>Proposition:</u> A maximally isotropic subspace $V \subset E$ is Lagrangian.

<u>Proof.</u> Suppose $V \subset V^\sigma$, but $V \subsetneq V^\sigma$, there exists $e \in V^\sigma \setminus V$. Let $v_1, v_2 \in V$, $\alpha_1, \alpha_2 \in \mathbb{R}$, then

$$\sigma(v_1 + \alpha_1 e, v_2 + \alpha_2 e) = 0$$

$$\forall v_1, v_2 \in V \Rightarrow V + \alpha e \text{ is isotropic.}$$

<u>Proposition:</u> If $V \subset E$ is isotropic, then V is Lagrangian $\Leftrightarrow \dim V = \frac{1}{2} \dim E$.

<u>Definition:</u> Let (S, σ) be a symplectic manifold, $\dim M = 2n$. A submanifold $\Sigma \subset S$ is

(i) Isotropic if $\forall p \in \Sigma$ $T_p\Sigma \subset (T_p\Sigma)^\sigma$

(ii) Involutive if $\forall p \in \Sigma$ $(T_p\Sigma)^\sigma \subset T_p\Sigma$

(iii) Lagrangian if $(T_p\Sigma)^\sigma = T_p\Sigma$

(iv) Symplectic if $(T_p\Sigma)^\sigma \cap T_p\Sigma = \{0\}$