Hamilton Jacobi Theory   Lesson 15   1

Let (S, σ) be a C^∞ symplectic manifold $\dim S = 2n$.

<u>Theorem</u>: Let $\Sigma \subset S$ be a C^∞ submanifold of codimension k. Through each $p \in \Sigma$ there passes a Lagrangian submanifold $\Lambda \subset \Sigma$ if and only if Σ is involutive.

<u>Lemma</u>: Let $\Sigma \subset S$ be locally defined by

$$\Sigma = \{ p \in S : f_1(p) = f_2(p) = \ldots = f_k(p) = 0 \}$$

Then Σ is involutive $\Longleftrightarrow$

$$\{ f_j, f_m \} = 0 \quad \text{on } \Sigma \text{ for } 1 \le j, m \le k.$$

<u>Proof</u>. If $p \in \Sigma$, $T_p \Sigma = \bigcap_{j=1}^{k} \operatorname{Ker} df_j(p)$

$V \in C^\infty(S, TS)$ is tangent to $\Sigma \Leftrightarrow$

$$V f_j = 0 \text{ on } \Sigma \quad 1 \le j \le k$$

But $V \in \mathrm{Ker}\, df(p) \Rightarrow df(V) = 0$ at p ②

$\Rightarrow \omega(H_f, V) = 0$ at $p \Rightarrow V \in (H_f(p))^\sigma$.

Hence $$T_p\Sigma = \bigcap_{j=1}^{k} H_{f_j}(p)^\sigma = \left(\sum_{j=1}^{k} \mathbb{R}\cdot H_{f_j}(m)\right)^\sigma.$$

Hence $(T_p\Sigma)^\sigma = \mathrm{Span}\{H_{f_i}(m) : 1 \le i \le k\}$

$\Rightarrow$) If $(T_p\Sigma)^\sigma \subset T_p\Sigma$, then

$$H_{f_j}(p) \in T_p\Sigma \quad j = 1, 2, \dots$$

$$df_k(H_{f_j}) = 0 \text{ at } p \quad j = 1, 2, \dots, k$$

$$\{f_j, f_k\} = 0 \text{ on } \Sigma.$$

$\Leftarrow$) If $\{f_j, f_k\} = 0$ on Σ, then H_{f_j} is tangent to $\Sigma \Rightarrow (T\Sigma)^\sigma \subset T\Sigma$.

Proof of the theorem.

$\Rightarrow$) Suppose that for any $p \in \Sigma$ there exists a Lagrangian submanifold $\Lambda \subset S$ such that $\Lambda \subset \Sigma$.

In this case $T_p\Lambda \subset T_p\Sigma \Rightarrow$

$$(T_p\Sigma)^\sigma \subset (T_p\Lambda)^\sigma = T_p\Lambda \subset T_p\Sigma$$

$\Rightarrow \Sigma$ is involutive.

$\Leftarrow$) Suppose Σ is involutive. Then locally, there exist $f_i \in C^\infty(U)$

$$\Sigma = \{ f_1 = f_2 = \dots = f_k = 0 \} \quad df_i \text{ lin ind}$$

$1 \le i, j \le k$.

$$\{ f_i, f_j \} = 0 \quad \text{on } \Sigma$$

$$d\{f_i, f_j\}(p) = \sum_{l=1}^{k} c_l(p)\, df_l(p)$$

$\Rightarrow$

$\cancel{H_{f_i}(f_j) = 0 \text{ on } \Sigma}$

$$H_{f_j}(p) \in T_p\Sigma$$

$$H_{\{f_i, f_j\}} = [H_{f_i}, H_{f_j}] = \sum \alpha_j H_{f_j}$$

$(T_p \Sigma)^\sigma = \text{Span}\{H_{f_j}\}$, hence

$(T_p \Sigma)^\sigma$ forms an integrable tangent system in the sense of Frobenius.

Codim $\Sigma = k$, hence dim $\Sigma = 2n - k$

Since $\dim T_p \Sigma + \dim (T_p \Sigma)^\sigma = 2n$,

$$\dim (T_p \Sigma)^\sigma = k$$

and hence $2n - k > k$.

<u>Claim:</u> One can choose a manifold M of codimension k in Σ such that in a neighborhood of $p \in \Sigma$, $T_x M \cap (T_x \Sigma)^\sigma = \{0\}$

$(T\Sigma)^p$ is spanned by H_{f_j} $1 \le j \le k$.

Pick M such that these vector fields are not tangent to it.

In this case $T_p M \subset T_p \Sigma$ and

hence $(T_p \Sigma)^\sigma \subset (T_p M)^\sigma$.

Since $T_p M \cap (T_p \Sigma)^\sigma = \{0\}$

$$\underbrace{(T_p M)^\sigma}_{\text{Dimension } 2k.} + \underbrace{T_p \Sigma}_{\text{Dimension } 2n-k.} = T_p S$$

dim of $(T_p M)^\sigma \cap T_p \Sigma = k$.

$= \dim (T_p \Sigma)^\sigma$

<u>Conclusion</u>: $(T_p M)^\sigma \cap T_p \Sigma = (T_p \Sigma)^\sigma$.

Since $T_p M \subset T_p \Sigma$, then

$$T_p M \cap (T_p M)^\sigma = T_p M \cap \left(T_p \Sigma \cap (T_p M)^\sigma\right)$$
$$= T_p M \cap (T_p \Sigma)^\sigma = \{0\}$$

So we conclude that

T_pM is a symplectic vector subspace of T_pS

Let $\Lambda_0 \subseteq M$ be a Lagrangian submanifold

Let $\Lambda = \{ \Phi_k^{t_k} \circ \dots \circ \Phi_1^{t_1}(p).\ p \in \Lambda_0 \}$

(1) Λ is a C^∞ submanifold of S by Frobenius' theorem.

(2) This manifold Λ has dimension n since

$$\dim M = 2n - 2k \quad , \quad \dim \Lambda_0 = n-k.$$

$$\dim \Lambda = n.$$

(3) Λ is involutive, Let $\Lambda_1 = \Phi_{H_{f_1}}^{t_1}(\Lambda_0)$

$$T_p(\Lambda_1) = T_p(\Lambda_0) + \mathbb{R} H_{f_1}.$$

$$(T_p(\Lambda_1))^\sigma = T_p(\Lambda_0)^\sigma \cap (H_{f_1})^\sigma =$$

$$= T_p(\Lambda_0)^\sigma \cap \operatorname{Ker} df_1$$

$$\supset \Lambda_0 + \mathbb{R}\cdot f_1 \supset T_p(\Lambda_1)$$

$T_p(\Lambda_1)$ is isotropic.

Repeat the argument until $T_p(\Lambda) = (T_p(\Lambda))^\sigma$.

Let $S = T^*X$; X a C^∞ submanifold

σ = canonical 2-form on S.

<u>Theorem</u>: Let $\Omega \subset T^*X$ be an open subset and let $f_j \in C^\infty(\Omega)$ $1 \leq j \leq k$ be such that

df_j $1 \leq j \leq k$ are linearly independent on

$$\Sigma = \{ (x,\xi) : f_1(x,\xi) = f_2(x,\xi) = \dots = f_k(x,\xi) = 0 \}.$$

I) If for every $(x_0,\xi_0) \in \Sigma$, there exists $\Phi(x) \in C^\infty(U_{x_0})$

U_{x_0} a neighborhood of x_0 such that

$$(HJ) \quad f_1(x, d_x\phi) = f_2(x, d_x\phi) = \dots = f_k(x, d_x\phi) = 0$$

Then (Inv) $\{f_i, f_j\} = 0$ on Σ and

$$(Li) \quad d_\xi f_1, d_\xi f_2, \dots, d_\xi f_k$$

are linearly independent on Σ.

Conversely Suppose Σ is involutive and (Li) holds., then for any $(x_0, \xi_0) \in \Sigma$.

and any submanifold $Q \subset X$ of dimension $n-k$ through x_0 that is transversal to $\pi_X H_{f_j}$, $1 \le j \le k$, and any $\psi \in C^\infty(Q)$ such that

$$d\psi|_{x_0} = \xi_0 \big|_{T_{x_0} Q}$$

There exists a neighborhood U_{x_0} of x_0 and a C^∞ function $\phi \in C^\infty(U_{x_0})$ such that

$$\phi = \psi \text{ on } Q.$$

and ϕ satisfies (HJ). Moreover, the graph of $d\phi$ in T^*X is a union of bicharacteristic strips. (invariant under the flow of H_{f_j} $1 \le j \le k$).

Proof:

$\Rightarrow$) Suppose Φ satisfies (HJ), then

the graph of $d\phi = \{(x,\xi) : \xi = d\phi\} = L$

is a Lagrangian submanifold, and is

contained in Σ. So by the previous

theorem $\{f_j : 1 \le j \le k\}$ are in involution

Let $$\begin{aligned} \pi : T^*X &\longrightarrow X \\ (x,\xi) &\longrightarrow x \end{aligned}$$

be the vector bundle projection. Since

L is a graph,

$$D_{(x,\xi)}\pi : T_{(x,\xi)}L \longrightarrow T_x X$$

is a diffeomorphism. But since $L \subset \Sigma$

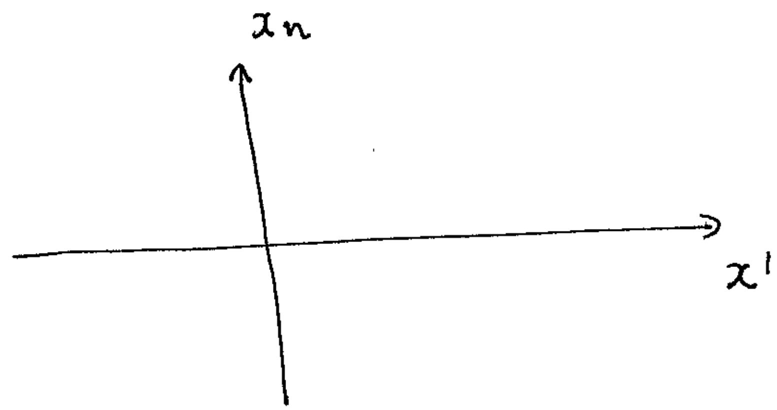
$$T_{(x,\xi)}L \supset (T_{x,\xi}\Sigma)^\sigma = \sum_{j=1}^{k} \mathbb{R}\, H_{f_j}$$

$D_{(x,\xi)}\pi\, H_{f_i}(x,\xi) \equiv d_\xi f_i(x,\xi)$ are linearly independent.

Assume $k=1$, $\pi(H_{f_1}) = \sum \frac{\partial f}{\partial \xi_\ell} \partial/\partial x_\ell \neq 0$

Example:

Suppose $\frac{\partial f}{\partial \xi_n}(x_0, \xi_0) \neq 0$ $\quad \xi_0 = (\xi_0', \xi_{0n})$



Take $\mathcal{U} = \{x_n = 0\}$, $\psi \in C^\infty(\mathbb{R}^{n-1})$, $\frac{\partial \psi}{\partial x'}(0) = \xi_0'$

By the implicit function theorem, there exists a unique $\lambda(x', \xi')$; $\lambda(0, \xi_0') = \xi_{0n}$

such that $f(x', 0, \xi', \lambda(x', \xi')) = 0$

Let

$$\Lambda' = \{(x, \xi) : x_n = 0, \ \xi' = d_{x'}\psi(x'), \ \xi_n = \lambda(x', \xi') \ x' \in \mathcal{U}\}.$$

Claim: Λ' is isotropic of dimension $n-1$.

Λ' is parametrized by x', so $\dim \Lambda' = n-1$.

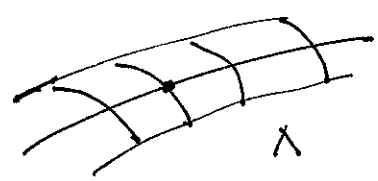
$\sigma|_{\Lambda'} = 0$ because $x_n = 0$ on Λ' and

$$\xi' = d_{x'} \varphi(x')$$

H_f is nowhere tangent to Λ', let

$\Lambda \subset f^{-1}(0)$ be the Lagrangian obtained by

taking the integral curves of H_f through Λ'



Notice that $\pi : \Lambda \longrightarrow X$ is a local diffeomorphism

near Λ'.

Theorem: Let $\Lambda \subset T^*X$ be a submanifold $\dim \Lambda = \dim X = n$. Suppose

$$\pi|_\Lambda : \Lambda \longrightarrow X$$

is a local diffeomorphism.

Then Λ is a Lagrangian $\Longleftrightarrow$

$\forall \; p \in \Lambda \quad \exists \quad \varphi \in C^\infty(U_{\pi(p)})$ such that

$$\Lambda = \{ (x, d\varphi(x)) : x \in U_{\pi(p)} \}.$$

Pf.: $\sigma = d\alpha$

$\sigma =$ canonical 2-form

$\alpha =$ canonical 1-form

$$d(\alpha|_\Lambda) = d\alpha|_\Lambda = \sigma|_\Lambda$$

But Λ is Lagrangian $\Longleftrightarrow \sigma|_\Lambda = 0 \Longleftrightarrow d\alpha|_\Lambda = 0$.

But

$\mathcal{Q} = \{ x_1 = x_2 = \dots = x_h = 0 \}$ $\qquad x = (x_1, \dots, x_h, x')$

$\psi \in C^\infty(\mathbb{R}^{n-h})$, $\dfrac{\partial \psi}{\partial x'}(0) = \xi_0' \Big|_{T_x \mathcal{Q}}$.

By the implicit function theorem.

$$f_j(0, x', \xi_1(x', \xi'), \dots, \xi_k(x', \xi') \; ; \; \xi') = 0$$

$$j = 1, 2, \dots, h.$$

$$\Lambda' = \{ (x, \xi) : x_u = 0 \quad \xi' = d_{x'} \psi' , \; \xi_j = \xi_j(x', \xi') , \; 1 \le j \le h , \; x \in U \}$$

Take the flow-out of Λ' by H_{f_1},

$$\Phi^{t_1}_{H_{f_1}} \circ H^{t_2}_{f_2} \circ \dots \circ H^{t_h}_{f_h} .$$