Parametrization of Lagrangian Submanifolds Lecture 17

Let $U \subset \mathbb{R}^n$ be open and $V \subset U \times \mathbb{R}^N$ be open.

Definition: A function $\varphi : V \longrightarrow \mathbb{R}$, $\varphi \in C^\infty$ is a non-degenerate phase function if

(i) $d_{(x,\theta)}\varphi \neq 0$ everywhere

(ii) If $\varphi'_\theta(x,\theta) = 0$, $d_{(x,\theta)}\dfrac{\partial\varphi}{\partial\theta_i}$ are linearly independent at (x,θ).

According to (i) and (ii)

$$C_\varphi = \{ (x,\theta) \in V : \varphi'_\theta(x,\theta) = 0 \}$$

is a closed submanifold of V of codimension N. (of dimension n).

Let

$$\begin{aligned} j : C_\varphi &\longrightarrow T^*X \\ (x,\theta) &\longmapsto (x, \varphi'_x(x,\theta)) \end{aligned}$$

<u>Lemma:</u> dj is injective at every point $(x,\xi) \in C_\varphi$.

<u>Proof:</u> Let $v = \underbrace{\sum_{j=1}^{n} a_j \partial_{x_j}}_{v_x} + \underbrace{\sum_{e=1}^{N} b_i \partial_{\theta_i}}_{v_\theta}$.

If v is tangent to C_φ, then $v\, \varphi'_{\theta_j}\big|_{C_\varphi} = 0$

for every $1 \le j \le N$.

$$\varphi''_{\theta x} v_x + \varphi''_{\theta\theta} v_\theta = 0$$

On the other hand $dj\, v = (v_x ; \varphi''_{xx} v_x + \varphi''_{x\theta} v_\theta)$

So if $dj\, v = 0 \Rightarrow v_x = 0$; $\varphi''_{x\theta} v_\theta = 0$

$\varphi_{\theta\theta} v_\theta = 0$.

$$\begin{pmatrix} \varphi''_{x\theta} \\ \varphi''_{\theta\theta} \end{pmatrix} v_\theta = 0 \Rightarrow v_\theta = 0$$

After shrinking V near any fixed point $(x_0, \theta_0) \in C_\varphi$ we may assume by the inverse function theorem that j is injective and that

$$\Lambda_\varphi = j(C_\varphi) \subset T^*X$$

is a C^∞ submanifold of dimension n.

~~Proposition~~ Proposition For V as above, $\Lambda\varphi$ is a Lagrangian submanifold.

~~Proposition~~ Proposition: Let $\Lambda \subset T^*X$ be a C^∞ Lagrangian submanifold. Then for every $(x_0, \xi_0) \in \Lambda$ one can find a non-degenerate phase function φ such that $\Lambda = \Lambda\varphi$ in a neighborhood of (x_0, ξ_0)

Proof. As we saw, there exists a neighborhood U_{x_0} of x_0 and coordinates x on U_{x_0} such that

$$\Lambda = \{ x = -\partial_\xi H \} \qquad H \in C^\infty(\mathbb{R}^n)$$

Let $\varphi(x, \theta) = \langle x, \theta \rangle + H(\theta)$

$$C_\varphi = \{ (x, \theta) : d_\theta \varphi = 0 \} = \{ (x, \theta) : x_j = -\frac{\partial H}{\partial \theta_j}, \ 1 \le j \le n \}$$

$$\Lambda\varphi = \{ (x, \xi) : \xi_j = \frac{\partial \varphi}{\partial x_j}, \ d_\theta \varphi = 0 \}$$

$$= \{ (x, \xi) : \xi_j = \theta_j \ ; \ x_j = -\frac{\partial H}{\partial \theta_j} \}$$

Conic Lagrangian Submanifolds

The manifold $T^*X \setminus 0 = \{ (x,\xi) \in T^*X , \xi \neq 0 \}$

has an $\mathbb{R}_+$ action defined by

$$(x,\xi) \longmapsto (x, \tau\xi), \quad \tau > 0$$

the $\mathbb{R}^*_+$ - action is generated by the radial vector field R. In local coordinates

$$R = \sum_{i=1}^{n} \xi_i \, \partial/\partial\xi_i$$

$$\Phi^\tau(x,\xi) = (x, \tau\xi) \quad ; \quad (\Phi^\tau)^* \varphi = \varphi(x, \tau\xi)$$

$$\frac{d}{d\tau}(\Phi^\tau)^*\varphi\Big|_{\tau=1} = \frac{d}{d\tau}\varphi(x,\tau\xi)\Big|_{\tau=1} = \sum_{i=1}^{n} \xi_i \partial_{\xi_i} \varphi(x,\xi)$$

Let α be the canonical 1-form and $\sigma = d\alpha$ be the canonical 2-form.

Proposition: $\sigma \lrcorner R = \alpha$.

Proof: In local coordinates $\sigma = \sum_{i=1}^{n} d\xi_i \wedge dx_i$

$$\sigma \lrcorner R = \sum_{j=1}^{n} \xi_j \, dx_j = \alpha.$$

<u>Proposition</u>: A closed submanifold $\Lambda \subset T^*X \setminus 0$ ~~is conic~~ $\dim \Lambda = n$ is a conic Lagrangian if and only if $\alpha = 0$ on Λ.

$\Rightarrow$) If Λ is conic, R is tangent to Λ.
If Λ is Lagrangian, then $\alpha = \sigma \lrcorner R = 0$ on Λ.

$\Leftarrow$) If $\alpha = 0$ on Λ, then $\sigma = d\alpha = 0$ on Λ.
Since $\dim \Lambda = n$, Λ is Lagrangian.

Since $\sigma \lrcorner R = \alpha = 0$, $R \in (T_p\Lambda)^{\sigma} = T_p\Lambda$
for every $p \in \Lambda$ $\Rightarrow$ R is tangent to Λ. Since Λ is closed, Λ has to be conic.

Parametrization of Conic Lagrangians

We shall assume that $\varphi : V \to \mathbb{R}$ is as above, but

$$\varphi(x, \lambda\theta) = \lambda\varphi(x,\theta) \quad , \lambda > 0.$$

φ is a homogeneous phase function.

<u>Proposition:</u> Let $\Lambda \subset T^*X \setminus 0$ be a C^∞ Conic Lagrangian submanifold. Then for every $(x_0, \xi_0) \in \Lambda$ one can find a non-degenerate homogeneous phase function φ such that

$$\Lambda = \Lambda_\varphi \quad \text{in a neighborhood}$$

of (x_0, ξ_0).

<u>VERY Important Remark:</u> $\varphi = 0$ on Λ_φ.

<u>Examples:</u> Let $\Sigma \subset X$ be a C^∞ submanifold of dimension $n-k$. The conormal bundle to Σ

$$N^*\Sigma = \{ (x, \xi) \in T^*X \setminus 0 : x \in \Sigma, \ \xi(v) = 0$$
$$\text{for } v \in T_x\Sigma \}$$

In local coordinates

$$\Sigma = \{ x \in X : \; x_1 = x_2 = \dots = x_k = 0 \}$$

$$\Lambda = N^* \Sigma = \{ x_1 = x_2 = \dots = x_k = 0, \quad \xi_{k+1} = \xi_{k+2} = \dots = \xi_n = 0 \}$$

Parametrization of Λ : $X = (x', x'')$; $\xi = (\xi', \xi'')$

$$x' = (x_1, \dots, x_k)$$

$$\varphi(x, \theta) = \langle x', \theta \rangle, \quad \theta \in \mathbb{R}^k$$

$$C_\varphi = \{ (x, \theta) : \; d_\theta \varphi = 0 \} = \{ (x,\theta) : \; x_1 = \dots = x_k = 0 \}$$

$$\Lambda_\varphi = \{ (x, \xi) : \; \xi_j = \frac{\partial \varphi}{\partial x_j}, \; d_\theta \varphi = 0 \}$$

$$= \{ (x, \xi) : \; x_1 = \dots = x_k = 0, \; \xi_j = \theta_j \; 1 \le j \le k$$
$$\xi_j = 0 \quad j = k+1, \dots, n \}.$$