Review of Pseudodifferential Operators, Lagrangian Distributions — Lecture 18

Let $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$, i.e. $a \in C^\infty$ and

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}.$$

$P \in \Psi^m(\mathbb{R}^n)$ if

$$Pu(x) = \frac{1}{(2\pi)^n}\int e^{i\langle x,\xi\rangle} a(x,\xi)\,\hat{u}(\xi)\,d\xi.$$

The Schwartz kernel of P is

$$K_P(x,y,\xi) = \frac{1}{(2\pi)^n}\int e^{i\langle x-y,\xi\rangle} a(x,\xi)\,d\xi$$

is a conormal distribution to the diagonal Δ.

and $K_P \in I^m(\mathbb{R}^n \times \mathbb{R}^n; \Delta)$.

Properties of K_P and P

(1) $P : H^s_{loc}(\mathbb{R}^n) \longrightarrow H^{s-m}_{loc}(\mathbb{R}^n)$

Ellipticity: We say that $a(x,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ is elliptic if there exists $R > 0$ such that

$$|a(x,\xi)| \geq C(1+|\xi|)^m \quad \text{for } |\xi| \geq R.$$

So if $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ there exists $b \in S^{-m}(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$ba - 1 \in S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n)$$

Set $b(x,\xi) = \dfrac{1}{a(x,\xi)}\chi(\xi)$ where $\chi(\xi) = 0$ for $|\xi| < R+1$

$\chi(\xi) = 1$ for $|\xi| > R+2$.

This can be done at the level of operators as well: Φ.

<u>Proposition</u>: If $P \in \Psi^m(\mathbb{R}^n)$ is elliptic (its symbol is elliptic), there exists $B \in \Psi^{-m}(\mathbb{R}^n)$ elliptic such that

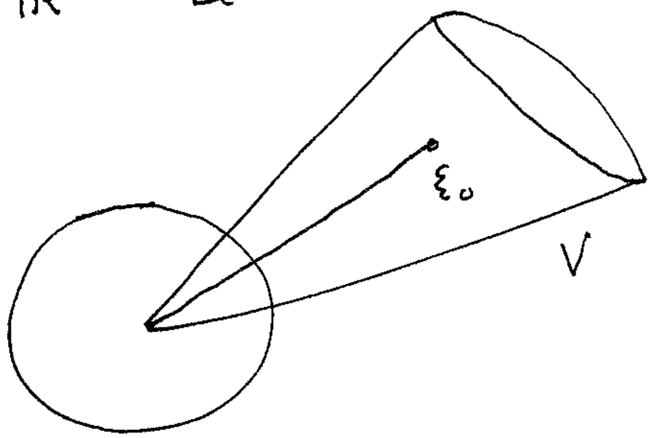
$$BP - I \in \Psi^{-\infty}(\mathbb{R}^n)$$
$$PB - I \in \Psi^{-\infty}(\mathbb{R}^n)$$

B is called a parametrix of P.

Microlocal Parametrices

Let $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \setminus 0$

Let $U \subset \mathbb{R}^n$ be a neighborhood of x_0 and $V \subset \mathbb{R}^n$ be a conic neighborhood of ξ_0



$$\left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| < \varepsilon$$

We say that $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ is elliptic at (x_0, ξ_0) if there exists a conic neighborhood $\xi_0 \in W \subset V$ and $x_0 \in \tilde{U} \subset U$ open such that

$$|a(x,\xi)| \geq C(1+|\xi|)^m$$

for $x \in \tilde{U}$ and $\xi \in W$; $|\xi| > R$..

Proposition: Let $P \in \Psi^m(\mathbb{R}^n)$ be elliptic at (x_0, ξ_0). Then there exists $B \in \Psi^{-m}(\mathbb{R}^n)$ such that $BP - I = E_1$ and $PB - I = E_2$ the symbols of E_1 and $E_2 \in S^{-\infty}(U_{x_0} \times W_{\xi_0})$

Where $x_0 \in U_{x_0}$ is an open subset and W_{ξ_0} (4)

is a conic open subset $\xi_0 \in W_\xi$.

The symbol calculus

Let $A \in \Psi^m(\mathbb{R}^n)$ $B \in \Psi^k(\mathbb{R}^n)$, then

$AB \in \Psi^{m+k}(\mathbb{R}^n)$. If

$$A u(x) = \frac{1}{(2\pi)^n} \int e^{i\langle x, \xi\rangle} a(x,\xi)\, \hat{u}(\xi)\, d\xi$$

$a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ we call

$a_m = [a] \in S^m / S^{m-1}$ the principal symbol of A.

If a_m is the principal symbol of A

and b_k is the principal symbol of B.

then the principal symbol of $AB = a_m b_k$.

The principal symbol of

$$[A, B] = \frac{1}{i}\{a_m, b_k\}.$$

Recall the Besov spaces $H^{\infty,\ell}_{loc}(\mathbb{R}^n)$ and $H^{\infty,\ell}_{loc}(X)$ when X is a C^∞ manifold: If $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\hat{u} \in L^2_{loc}$,

$$\Pi_j u(x) = (2\pi)^{-n} \int_{X_j} e^{i\langle x, \xi\rangle} \hat{u}(\xi)\, d\xi$$

$$X_j = \{ \xi \in \mathbb{R}^n : 2^{j-2} < |\xi| < 2^j \} \quad j > 0$$

$$X_0 = \{ \xi \in \mathbb{R}^n : 0 < |\xi| < 1/2 \}.$$

Essentially $\widehat{\Pi_j u}(\xi) = \mathbb{1}_{X_j} \hat{u}(\xi)$ where $\mathbb{1}_{X_j}$ is the indicator function of X_j

$$\| \Pi_j u \|_s^2 = \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{\Pi_j u}(\xi)|^2\, d\xi.$$ is the Sobolev norm of $\Pi_j u$.

$u \in H^{\infty, s}_{loc}$ if $\sup_j \| \Pi_j u \|_s < \infty$.

The space $H^{\infty,s}_{loc}(X)$ can easily be defined by using coordinate patches (Exercise)

Definition: Let X be a C^∞ manifold and let $\Lambda \subset T^*X \setminus 0$ be a C^∞, closed, conic Lagrangian submanifold. Let E be a vector bundle over X. Then the space $I^m(X;\Lambda,E)$ of Lagrangian distribution sections of E of order m is the set of $u \in \mathcal{D}'(X,E)$ such that

$$L_1 L_2 \cdots L_N u \in H^{\infty,-m-n/4}_{loc}(X,E)$$

for all N and all properly supported $L_j \in \Psi^1(X;E)$ with principal symbol L_j^0 vanishing on Λ.

Example: Let $\Sigma \subset X$ be a C^∞ submanifold of codimension k. The conormal bundle to Σ is defined to be

$$N^*\Sigma = \{ (x,\xi) \in T^*X \setminus 0 : \ \xi(v) = 0 \text{ for all } v \in T_x\Sigma \}$$