Lecture 19: Lagrangian Distributions Continuation

Prop (Microlocalization) If $u \in I^m(X, \Lambda)$, then

(i) $WF\, u \subset \Lambda$

(ii) If $A \in \Psi^0(X)$ then $Au \in I^m(X, \Lambda)$

Conversely: $u \in I^m(X, \Lambda) \iff \forall (x_0, \xi_0) \in T^*X \setminus 0$ one can find $A \in \Psi^0(X)$ properly supported, elliptic at (x_0, ξ_0) such that $Au \in I^m(X, \Lambda)$.

Proof: (i) If $(x_0, \xi_0) \notin \Lambda$, one can choose $L_1, \dots, L_N$ non-characteristic at (x_0, ξ_0) and conclude that $u \in H^s(\Gamma)$ on a conic neighborhood of (x_0, ξ_0)

$\Rightarrow WF u \cap \Gamma = \emptyset$.

(ii) Let $A \in \Psi^0(X)$ and $u \in I^m(X, \Lambda)$

$$L_1 Au = AL_1 u + [L_1, A]u \in H^{\infty, -m-n/4}_{loc}(X)$$

$$L_2 L_1 Au = L_2 A L_1 u + L_2 [L_1, A] u =$$

$$= \underbrace{A L_2 L_1 u + [L_2, A] L_1 u}_{\in H^{\infty, -m-n/4} \text{ by continuity}} + \underbrace{L_2 [L_1, A] u}_{\in H^{\infty, -m-n/4} \text{ by previous step}}$$

By induction one gets that $L_1 L_2 \ldots L_N Au \in H^{\infty, -m-n/4}$ (2)

$\Leftarrow$) Suppose $\forall\ (x_0, \xi_0) \in T^*X \setminus 0$ $\exists$ $A \in \Psi^0(X)$ elliptic at (x_0, ξ_0) such that $Au \in I^m(X, \Lambda)$.

$\exists$ conic neighborhood Γ of (x_0, ξ_0) and B elliptic on Γ such that

$$BA - I \in \Psi^{-\infty} \text{ on } \Gamma.$$

Since $Au \in I^m(X, \Lambda)$, it follows that

$$(BA - I)u \in I^m(X, \Lambda) \quad \text{on } \Gamma.$$

$$u \in I^m(X, \Lambda) \quad \text{on } \Gamma.$$

$$\text{of } T^*X \setminus 0.$$

Use a partition of unity

In view of this proposition we may work to characterize microlocally near $(x_0, \xi_0) \in T^*X \setminus 0$ $u \in I^m(X, \Lambda)$. We may assume $X = \mathbb{R}^n$.

and

$$\Lambda = \left\{ (x, \xi) : x_j = -\frac{\partial H}{\partial \xi_j}, \ \xi \in \mathbb{R}^n \setminus 0 \right\}$$

<u>Proposition:</u> If $u \in I^m_{comp}(\mathbb{R}^n, \Lambda)$, with

$\Lambda = \{(x,\xi) : x_j = \frac{\partial H}{\partial \xi_j}, \ \xi \in \mathbb{R}^n \setminus 0\}$, $H(\xi)$

homogeneous of degree 1, then

$$\hat{u}(\xi) = e^{-iH(\xi)} v(\xi) \qquad |\xi| > 1$$

where $v \in S^{m-n/4}(\mathbb{R}^n)$.

Conversely, if $v \in S^{m-n/4}(\mathbb{R}^n)$, then

$$\mathcal{F}\left(e^{-iH(\xi)} v(\xi)\right) \in I^m(\mathbb{R}^n, \Lambda).$$

<u>Proof:</u> Let $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi = 1$ near $\xi = 0$.

Let $h = (1 - \chi(\xi)) H(\xi)$; $H_0 \in C^\infty$ and.

$h(\lambda \xi) = \lambda H_0(\xi)$ if $|\xi| \gg 1$. So $h \in S^1$

and h as principal symbol H. So it is enough to prove the statement for H replaced by h. Set $h_j(\xi) = \frac{\partial h}{\partial \xi_j}$

Let $h(D)u = \frac{1}{(2\pi)^n} \int e^{i\langle x,\xi\rangle} \hat{h}(\xi)\hat{u}(\xi)\, d\xi.$ (4)

If $u \in I^m(\mathbb{R}^n, \Lambda)$.

$$D^\beta \prod (x_j - h_j(D))^{\alpha_j} u \in H^{\infty, -m-n/4}(\mathbb{R}^n)$$

Provided $|\beta| = |\alpha|$. Notice that

$[x_j - h_j(D) ; D_k] = i\,\delta_{jk}$ so commuting.

the factors D^β we obtain a sum of products of operators of the form $(x_j - h_j(D))D_k$.

Recall the definition of $H^{\infty, -m-n/4}(\mathbb{R}^n)$.

$$\int_{R/2 < |\xi| < 2R} \left| \xi^\beta \prod (+D_j - h_j(\xi))^{\alpha_j} \hat{u}(\xi) \right|^2 d\xi \leq C_\alpha R^{2(m+n/4)}$$

$R > 1$; $|\beta| = |\alpha|$.

$$\hat{u}(\xi) = e^{-ih(\xi)} v(\xi)$$

$$D_{\xi_j} \hat{u} = e^{-ih(\xi)} (-\partial_j h + D_j) v.$$

$$(D_{\xi_j} + h_j)\hat{u} = e^{-ih(\xi)} D_j v$$

This means that

$$\int_{R/2 < |\xi| < 2R} |\xi|^{2|\alpha|} \, |D^\alpha v(\xi)|^2 \, d\xi \le C_\alpha R^{2(m+n/4)}$$

Exercise: Review the proofs of the theorems for Canonical Distributions. $v \in S^{m-n/4}(\mathbb{R}^n)$

· The argument can be reversed to prove the converse of the statement. Just need to observe that if $p(x,\xi)$ is homogeneous of degree 1 and $p = 0$ on Λ

$$p = \sum_{j=1}^{n} q_j(x,\xi)\,(x_j - h_j(\xi))$$