Lesson 21: Phase Functions and Oscillatory Integrals

Definition: Let $X \subset \mathbb{R}^n$ be open, A C^∞ function

$\varphi : X \times (\mathbb{R}^N \setminus 0) \longrightarrow \mathbb{R}$ is a phase function

if

(i) $$d\varphi = \sum_{j=1}^{n} \frac{\partial \varphi}{\partial x_j} dx_j + \sum_{m=1}^{N} \frac{\partial \varphi}{\partial \theta_j} d\theta_j \neq 0$$

(ii) $$\varphi(x, \lambda\theta) = \lambda \varphi(x, \theta), \quad \lambda > 0$$

Let $a \in S^m(X \times \mathbb{R}^N)$ be a symbol, the integral

$$I(a,\varphi)(x) = \int e^{i\varphi(x,\theta)} a(x,\theta)\, d\theta$$

is well defined if $m < -N$

We want to define $I(a,\varphi)(x)$ for $a \in S^m(X \times \mathbb{R}^N)$ for any m. We need to define this in the sense of distribution.

$$I(a,\varphi)(x)\, u(x) = \int e^{i\varphi(x,\theta)} a(x,\theta)\, d\theta \quad u(x)\, dx.$$

Idea, let L be operator whose adjoint is defined by

$$L^t = \frac{1-\chi(\theta)}{i\,\Phi(x,\theta)} \left(\sum_{j=1}^{N} |\theta|^2 \overline{\frac{\partial\varphi}{\partial\theta_j}} \frac{\partial}{\partial\theta_j} + \sum_{j=1}^{n} \overline{\frac{\partial\varphi}{\partial x_j}} \frac{\partial}{\partial x_j} \right) + \chi(\theta)$$

where $\Phi = \sum_{j=1}^{n} \overline{\frac{\partial\varphi}{\partial x_j}} \frac{\partial\varphi}{\partial x_j} + |\theta|^2 \sum_{j=1}^{N} \left|\frac{\partial\varphi}{\partial\theta_j}\right|^2$

$$\Phi(x,\lambda\theta) = \lambda^2\, \Phi(x,\theta) \quad , \theta \neq 0, \quad \lambda > 0$$

<u>Exercise</u>: <u>Verify that</u>: (i) $L^t e^{i\varphi} = e^{i\varphi}$

(ii) $L = (L^t)^t = \sum_{j=1}^{N} a_j(x,\theta) \frac{\partial}{\partial\theta_j} + \sum_{j=1}^{n} b_j(x,\theta) \frac{\partial}{\partial x_j} + c(x,\theta)$

where $a_j \in S^0(X, \mathbb{R}^N)$; $b_j, c_j \in S^{-1}(X \times \mathbb{R}^N)$.

$$\langle I(a,\varphi), u \rangle = \iint e^{i\varphi(x,\theta)} a(x,\theta) u(x)\, dx\, d\theta.$$

$$= \iint (L^t)^k e^{i\varphi(x,\theta)} a(x,\theta) u(x)\, dx\, d\theta.$$

$$= \iint e^{i\varphi(x,\theta)} \underbrace{L^k \left(a(x,\theta) u(x) \right)}_{\in S^{m-k}(X \times \mathbb{R}^N)} dx\, d\theta$$

Just pick k large enough : $m-k < -N$

The integral converges. This is independent of k, provided $m-k < -N$.

$I(a,\varphi) \in \mathcal{D}'(X)$.