Lecture 22 Lagrangian Distributions - Continuation

Let X be a C^∞ manifold and let $\Lambda \subset T^*X \setminus 0$ be a conic, closed, C^∞ Lagrangian submanifold.

Proposition: Let $\varphi(x,\theta)$ be a non-degenerate phase function in an open conic neighborhood of $(x_0,\theta_0) \in \mathbb{R}^n \times (\mathbb{R}^N \setminus 0)$ which parametrizes Λ in a neighborhood of (x_0,ξ_0), $\varphi'_\theta(x_0,\theta_0) = 0$; $\xi_0 = \varphi'_x(x_0,\theta_0)$.

If $a \in S^{m+(n-2N)/4}(\mathbb{R}^n \times \mathbb{R}^N)$ has support in the interior of a sufficiently small conic neighborhood Γ of (x_0,θ_0) and if

$$(*) \quad u(x) = \frac{1}{(2\pi)^{(n+2N)/4}} \int e^{i\varphi(x,\theta)} a(x,\theta)\, d\theta$$

then $u \in I^m_{comp}(\mathbb{R}^n; \Lambda)$. Moreover,

if

(A) $\quad \Lambda = \{ (x,\xi) : x_j = \dfrac{\partial H}{\partial \xi_j} \}$

then for $|\xi| > 1$

$$e^{iH(\xi)}\hat{u}(\xi) - (2\pi)^{n/4} a(x,\theta)\, e^{i\frac{\pi}{4}\operatorname{sgn}\Phi} |\det \Phi|^{-1/2} \in S^{m-n/4-1} \qquad (**)$$

where (x,θ) is determined by $\varphi'_\theta(x,\theta)=0$ and $\xi = \varphi'_x(x,\theta)$, $a(x,\theta)=0$ if there is no such point in Γ, and

$$\Phi = \begin{pmatrix} \varphi''_{xx} & \varphi''_{x\theta} \\ \varphi''_{\theta x} & \varphi''_{\theta\theta} \end{pmatrix}.$$

Conversely, every $u \in I^m(X,\Lambda)$ with $WF\,u$ in a small conic neighborhood of (x_0,ξ_0) can, modulo C^∞, be written in the form $(*)$.

<u>Proof:</u> Let $u(x)$ be given by $(*)$. Then

$$e^{iH(\xi)}\hat{u}(\xi) = (2\pi)^{-(n+2N)/4} \iint e^{i(\varphi(x,\theta) - \langle x,\xi\rangle + H(\xi))} a(x,\theta)\, dx\, d\theta$$

The goal is to show that $e^{iH(\xi)}\hat{u}(\xi) \in S^{m-n/4}$ and that $(**)$ holds

The phase of the integral is given by

$$\psi(x,\theta,\xi) = \varphi(x,\theta) - \langle x,\xi\rangle + H(\xi)$$

and the integration is on x and θ, so we have to consider the critical points of $\psi(x,\theta,\xi)$ in (x,θ), and these are:

$$(1)\qquad \frac{\partial\psi}{\partial x_j} = \frac{\partial\varphi}{\partial x_j} - \xi_j = 0 \quad ; \quad \frac{\partial\psi}{\partial\theta_\ell} = \frac{\partial\varphi}{\partial\theta_\ell} = 0$$

$$1 \le j \le n \qquad\qquad 1 \le \ell \le N$$

Since φ parametrizes Λ, ~~the~~ (1) implies that $(x,\xi) \in \Lambda$. On the other hand, in view of (A)

$$x_j = \frac{\partial H}{\partial \xi_j}$$

Observe that for each ξ, there exists a unique (x,θ) satisfying (1). The map

$$\{(x,\theta) : d_\theta\varphi = 0\} = C \longmapsto \Lambda = \{(x,\xi) : \xi = \frac{\partial\varphi}{\partial x}, d_\theta\varphi = 0\}$$

is a diffeomorphism. But because of (A) the projection $(x,\xi) \in \Lambda \to \xi = \varphi_x'$ is a diffeomorphism.

A vector field

$$V = \sum_{j=1}^{n} a_j(x,\theta)\partial_{x_j} + \sum_{\ell=1}^{N} b_\ell(x,\theta)\partial_{\theta_\ell}$$

is tangent to C if

$$V\left(\frac{\partial\varphi}{\partial\theta_m}\right) = 0 \quad \text{when} \quad \frac{\partial\varphi}{\partial\theta_k} = 0 \quad k = 1,2,\dots,N$$

If $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix}$, this means

that $\varphi_{\theta x} a + \varphi_{\theta\theta} b = 0$

The projection

$$\pi : C \longrightarrow \mathbb{R}^n$$
$$(x,\theta) \longmapsto \varphi_x'$$

is a diffeomorphism and $d\pi(V) =$

$$\varphi_{xx} a + \varphi_{\theta x} b$$

So we conclude that the kernel

of $\begin{pmatrix} \varphi_{xx} & \varphi_{x\theta} \\ \varphi_{\theta x} & \varphi_{\theta\theta} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$

$\Rightarrow$ $\cancel{a}\cdot\begin{pmatrix} a \\ b \end{pmatrix} = 0$

and so $\Phi = \begin{pmatrix} \varphi_{xx} & \varphi_{x\theta} \\ \varphi_{\theta x} & \varphi_{\theta\theta} \end{pmatrix}$ is non-singular (5)

If we ~~multiply~~ divide the first n rows by $|\theta|$ and multiply the last N columns by $|\theta|$ we obtain the matrix

$$\tilde{\Phi} = \begin{pmatrix} \varphi_{xx}/|\theta| & \varphi_{x\theta} \\ \varphi_{\theta x} & \varphi_{\theta\theta}|\theta| \end{pmatrix}$$

which is homogeneous of degree 0. therefore $\det \Phi$ is homogeneous of degree $n - N$ in θ.

Therefore

$$a(x,\theta)\,|\det \Phi|^{-1/2} \in S^{m-n/4},$$

in a conic neighborhood of C_i

We want to show this remains true for the restriction to C regarded as a function of ξ. We need the following Lemma:

Lemma: Let $\Gamma_j \subset \mathbb{R}^{n_j} \times (\mathbb{R}^{N_j} \setminus 0)$ $j = 1, 2$ be open cones and let

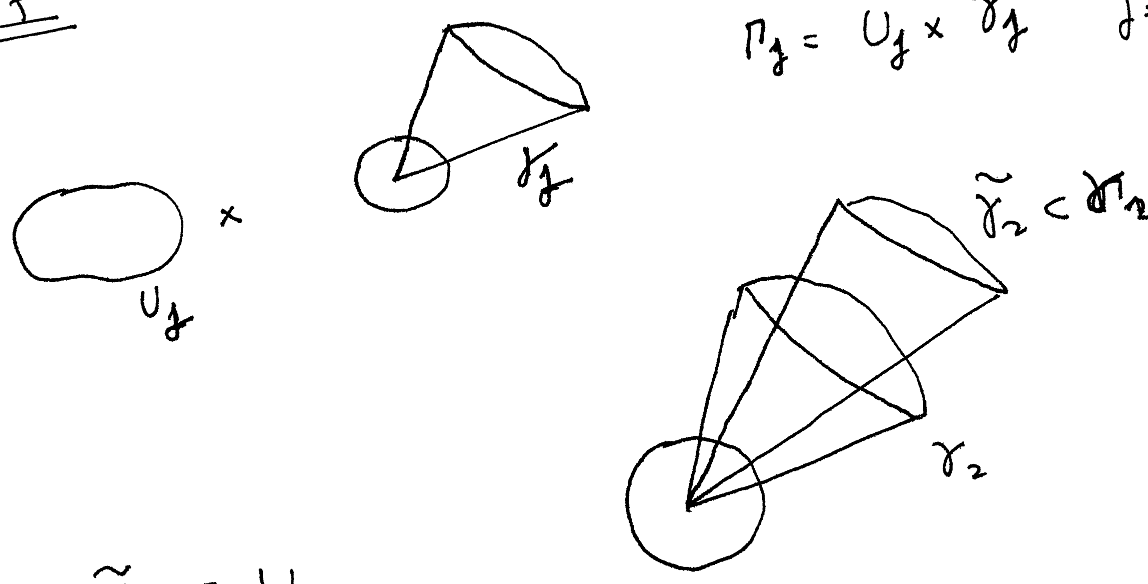
$$\psi : \Gamma_1 \longrightarrow \Gamma_2$$

be a C^∞ proper map commuting with multiplication by positive scalars in the second variable.

If $a \in S^m(\mathbb{R}^{n_2} \times \mathbb{R}^{N_2})$ has support in the interior of a compactly based cone $\subset \Gamma_2$ then $a \circ \psi \in S^m(\mathbb{R}^{n_1} \times \mathbb{R}^{N_1})$ if the composition is defined as 0 outside Γ_1.

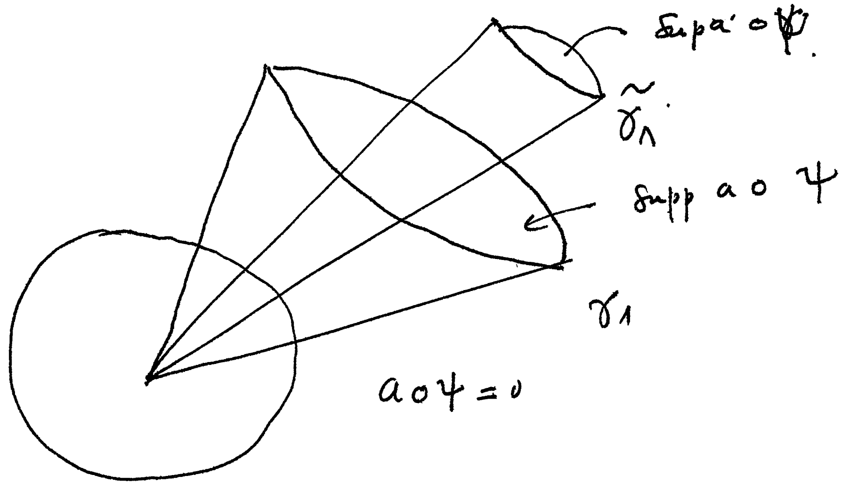
Proof:

$\Gamma_j = U_j \times \gamma_j \quad j = 1, 2.$



$\tilde{U}_2 \subset U_2$

$\operatorname{supp} a \subset \tilde{U}_2 \times \tilde{\gamma}_2 = \Gamma_2 \subset\subset \Gamma_2.$



<u>Proof:</u> The support of $a \subset \tilde{U}_2 \times \tilde{\gamma}_2 = \tilde{\Pi}_2$

If $\psi(x, \xi) = (y, \eta) \in \tilde{\Gamma}_2 \Rightarrow$

$$|\xi|/c \leq |\eta| \leq c|\xi|$$

Since $a \in S^m(\mathbb{R}^{n_2} \times \mathbb{R}^{N_2})$

$$|D^\alpha_{y,\eta}\, a(y, t\eta)| \leq C_\alpha t^m$$

provided $1/c \leq |\eta| \leq c$.

Since $(a \circ \psi)(x, t\xi) = (a(\cdot, t\,\cdot) \circ \psi)(x, \xi)$ by the homogeneity of ψ, we obtain

$$|D^\alpha_{x\xi}(a \circ \psi)(x, t\xi)| \leq C_\alpha t^m \quad \text{for } |\xi| = 1.$$

This proves the lemma.

By assumption $a(x,\theta)$ is supported in

$$\overline{B(x_0,\delta)} = \{x : |x - x_0| \le \delta\} \qquad \overline{\Gamma_\varepsilon} = \left\{\theta : \left|\frac{\theta}{|\theta|} - \frac{\theta_0}{|\theta_0|}\right| \le \varepsilon\right\}$$

and the map

$$C = \{(x,\theta) : d_\theta\varphi = 0\} \xrightarrow{\psi} \Lambda = \left\{(x,\xi) : \xi_i = \frac{\partial\varphi}{\partial x_i},\ d_\theta\varphi = 0\right\}$$

Commutes with dilation in the 2nd variable.

So the lemma applies to the map

$$a(x,\theta)\,|\det\Phi|^{-1/2}\Big|_C \longrightarrow$$

$$a(x(x,\xi);\theta(x,\xi))\,|\det\Phi(x(x,\xi);\theta(x,\xi))| \text{ on } \Lambda.$$

□

Now we return to the oscillatory integral

$$\int e^{i(\varphi(x,\theta) - \langle x,\xi\rangle)}\, a(x,\theta)\, dx.$$

$a(x,\theta)$ supported in $\overline{B(x_0,\delta)} \times \overline{\Gamma_\varepsilon}$.

Set $$f_{(\xi,\theta)}(x) = (\varphi(x,\theta) - \langle x,\xi\rangle)(|\xi|+|\theta|)^{-1}$$

This is homogeneous of degree 0 in (ξ, θ)

$$f'(x) = \left(\varphi'_x(x,\theta) - \xi\right)\left(|\xi| + |\theta|\right)^{-1}$$

$$|f'(x)| \geq \left(|\xi| - C_1|\theta|\right)\left(|\xi| + |\theta|\right)^{-1} \geq 1/2$$

if $\frac{|\theta|}{|\xi|}$ is small.

Since $\varphi'_x(x_0, \theta_0) = \xi_0 \neq 0$, then

$$\left|\varphi'_x\left(x, \frac{\theta}{|\theta|}\right)\right| \geq \tilde{C} \text{ if } \delta \text{ and } \varepsilon$$

are small enough.

So

$$|f'(x)| \geq \left(\tilde{C}|\theta| - |\xi|\right)\left(|\xi| + |\theta|\right)^{-1} \geq 1/2$$

if $|\xi|/|\theta|$ is small.

So we can apply stationary phase to the integral in the region $|\theta| \geq C|\xi|$ or $|\xi| \geq C|\theta|$.

$$\left|\int e^{i(\varphi(x,\theta) - \langle x, \xi\rangle)} a(x,\theta)\, dx\right| \leq C_N\left(|\xi| + |\theta|\right)^{-N}$$

$\forall N$.

Now let $\chi \in C_0^\infty(\mathbb{R}^N \setminus 0)$ be such that

$$\chi(\theta) = 1 \quad \text{if} \quad 1/C < |\theta| < C$$

So we conclude that

$$e^{iH(\xi)}\hat{u}(\xi) - (2\pi)^{-(n+2N)/4} \iint e^{i(\varphi(x,\theta)+H(\xi)-\langle x,\xi\rangle)} \chi\left(\frac{\theta}{|\xi|}\right) a(x,\theta)\,dx\,d\theta$$

$$= O(|\xi|^{-N}) \quad \forall N.$$

Set $|\xi| = t$ and $\theta/t = \omega$, $\eta = \xi/|\xi|$

$$U(\xi) = (2\pi)^{-(n+2N)/4} \iint e^{i(\varphi(x,\theta)+H(\xi)-\langle x,\xi\rangle)} \chi\left(\frac{\theta}{|\xi|}\right) a(x,\theta)\,dx\,d\theta$$

$$= (2\pi)^{-(n+2N)/4} \iint e^{it(\varphi(x,\omega)+H(\eta)-\langle x,\eta\rangle)} \chi(\omega)\, a(x,t\omega)\,dx\,t^N d\omega.$$

The phase has a unique critical pt on the support of the integrand and is given by $\varphi'_x(x,\omega) - \eta = 0$; $\varphi'_\omega(x,\omega) = 0$.

Since φ is homogeneous of degree 1.

$$\varphi(x, t\theta) = t\,\varphi(x,\theta) \quad \text{and hence} \quad \varphi(x,\theta) = \langle \theta, \varphi'_\theta(x, t\theta)\rangle$$

Set $t = 1$. $\varphi(x,\theta) = \langle \theta, \varphi'_\theta(x,\theta)\rangle$ so

$$\varphi(x,\theta) = 0 \text{ if } \varphi'_\theta(x,\theta) = 0.$$

Similarly, since $x = \dfrac{\partial H}{\partial \xi_i}$;

$$\langle x, \eta\rangle = \left\langle \frac{\partial H}{\partial \eta_i}, \eta \right\rangle = H(\eta)$$

So $\langle x, \eta\rangle - H(\eta) = 0$

and hence the phase $= 0$ on the set of critical points. Hence, applying the stationary phase theorem, and since $X = 1$ at the critical point.

$$U\cancel{(\xi)} = \cancel{(2\pi)^{n/4}\, a(x,\theta)}\; e^{i\frac{\pi}{4}\operatorname{sgn}\Phi(x,\theta)}\; \cancel{|\det\Phi|}^{-1/2}$$

$$\mathcal{U}(\xi) = (2\pi)^{n/4}\, a(x, t\omega)\, t^{(N-n)/2}\, e^{i\pi/4 \operatorname{sgn} \Phi(x,\omega)}$$
$$|\det \Phi(x, \omega)|^{-1/2}.$$

$$= (2\pi)^{n/4}\, a(x, \theta)\, e^{i\pi/4 \operatorname{sgn} \Phi(x,\theta)}\, |\det \Phi(x,\theta)|^{-1/2}$$

$+ \cdots$

To prove the converse, consider $u \in I^m(X, \Lambda)$ with $v(\xi) = e^{iH(\xi)}\hat{u}(\xi) \in S^{m-n/4}$ having support in a small conic neighborhood of ξ_0.

Choose a C^∞ map

$$(x, \theta) \longmapsto \psi(x, \theta) \in \mathbb{R}^n \setminus 0$$

in a conic neighborhood of (x_0, θ_0) such that ψ is homogeneous of degree 1

$$\psi(x, \theta) = \frac{\partial \varphi}{\partial x} \quad \text{when} \quad \frac{\partial \varphi}{\partial \theta} = 0.$$

Let

$$a_0(x,\theta) = (2\pi)^{-n/4}\, v(\psi(x,\theta))\, e^{-i\pi i \operatorname{sgn} \Phi} |\det \Phi|^{1/2}$$

$$\in S^{m+(n-2N)/4}$$

near C and define u_0 by $(*)$ with a replaced by a_0. Then

$$u - u_0 \in I^{m-1}.$$

Repeat the argument to find a sequence

$$a_j \in S^{m+(n-2N)/4-j}$$

such that

$$u - \sum_{k=0}^{j} u_k \in I^{m-j-1}.$$

If $a \sim \sum a_j$ then $(*)$ is valid modulo C^∞ $\square$