# Lesson 24 : The Calculus of F.I.O

Let $X$ and $Y$ be $C^\infty$ manifolds and let $\sigma_X$ and $\sigma_Y$ denote the respective symplectic forms on $T^*X$ and $T^*Y$ respectively.

**Exercise:** $T^*X \times T^*Y$ is a symplectic manifold with either one of the forms:

$$\sigma_+ = \pi_1^* \sigma_X + \pi_2^* \sigma_Y$$

or

$$\sigma_- = \pi_1^* \sigma_X - \pi_2^* \sigma_Y$$

$$\begin{tikzcd} & T^*X \times T^*Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ T^*X & & T^*Y \end{tikzcd}$$

In local coordinates:

$$\sigma_\pm = \sum_{j=1}^{n} d\xi_j \wedge dx_j \pm \sum_{\ell=1}^{m} d\eta_\ell \wedge dy_\ell$$

A Lagrangian manifold $C \subset T^*X \times T^*Y$ with respect to $\sigma_-$ is called a canonical relation

One can think of a set $C \subset T^*X \times T^*Y$ as a relation from $T^*Y$ to $T^*X$ defined by the map

$$(y, \eta) \longrightarrow (x, \xi)$$

such that $(y, \eta, x, \xi) \in C$.

<u>Example</u>: If $\chi : T^*X \longrightarrow T^*Y$ is a symplectomorphism: the graph of $\chi$ is a canonical transformation from $T^*Y$ to $T^*X$

<u>Pf</u>: $\sigma = \sigma_X - \sigma_Y$ and $\sigma\big|_{\Gamma(\chi)} = \sigma_X - \chi^* \sigma_Y = 0$

If $C$ is a canonical relation, then

$$C' = \{ (x, \xi, y, -\eta) : (x, \xi, y, \eta) \in C \}$$

is called the twisted relation

is a Lagrangian with respect to $\sigma_+$.

<u>Definition:</u> Let $C$ be a ~~can~~ homogeneous canonical relation from $T^*Y \setminus 0$ to $T^*X \setminus 0$ which is closed in $T^*(X \times Y) \setminus 0$. Then the operators whose Schwartz kernels belong to $I^m(X \times Y; C', \Omega^{1/2}_{X \times Y})$ are called Fourier integral operators of order $m$ with respect to $C'$.

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<u>Adjoints:</u> Let $A: C_0^\infty(X) \to C_0^\infty(Y)$ be linear and continuous and let $K(x,y)$ denote the kernel of $A$. Then

$$Au(y) = \int K(y,x)\, u(x)\, dx$$

$$K \in \mathcal{D}'(Y \times X)$$

The adjoint of the operator is

$$A^* u(x) = \int \overline{K(x,y)}\, u(y)\, dy$$

If

$$K(y,x) = \int e^{i\varphi(y,x,\theta)}\, a(y,x,\theta)\, d\theta.$$

with $\varphi$ parametrizing $\Lambda \subset (T^*Y \setminus 0) \times (T^*X \setminus 0)$

Then $C = \{ \varphi'_\theta = 0 \}$

$$\Lambda = \{ (y,x,\eta,\xi) : \eta = \varphi'_y ; \ \xi = \varphi'_x \}$$

The adjoint kernel

$$\overline{K(x,y)} = \int e^{-i\varphi(x,y,\theta)}\, \overline{a}(y,x,\theta)\, d\theta$$

The phase now parametrizes

$$\Lambda^{-1} \in (T^*X \setminus 0) \times (T^*Y \setminus 0)$$

$$\Lambda^{-1} = \{ (x,y,\xi,\eta) : (y,x,\eta,\xi) \in \Lambda \}$$

<u>Theorem:</u> Let $C$ be a canonical relation from $T^*Y \backslash 0$ to $T^*X \backslash 0$ which is closed in $T^*(X \times Y) \backslash 0$. ~~and~~ If $A \in I^m(X \times Y, C', \Omega^{1/2}_{X \times Y})$ then $A^* \in I^m(Y \times X; (C^{-1})'; \Omega^{1/2}_{X \times Y})$

If $a \in S^{m+n/4}(C; \Omega^{1/2}_C \otimes M_C)$ is a principal symbol for $A$, $n = \dim(X+Y)$, then $S^* a^* \in S^{m+n/4}(C^{-1}, \Omega^{1/2}_{C^{-1}} \otimes M_{C^{-1}})$,

$$S : Y \times X \longrightarrow X \times Y$$
$$(y, x) \longmapsto (x, y).$$

<u>Composition:</u> If $K_1(y, x)$, $K_2(z, y)$ are kernels of $A_1 : C^\infty_0(X) \to C^\infty(Y)$ and $A_2 : C^\infty_0(Y) \longrightarrow C^\infty_0(Z)$, $A_2 A_1 : C^\infty_0(Y) \to C^\infty_0(Z)$

$$A_2 A_1 u(z) = \int\int K_2(z, y) K_1(y, x)\, u(x)\, dy\, dx$$

$$= \int K(z, x)\, u(x)\, dx$$

$$K(z,x) = \int_X k_2(z,y)\, k_1(y,x)\, dy$$

We need to assume this integral is well defined.

Let us ~~see~~ try to see what is involved in this operation: at the level of support of the distributions $k_1$ and $k_2$.

$$\mathrm{Supp}\, k_1 \subset Y \times X; \quad \mathrm{Supp}\, k_2 \subset Z \times Y$$

$$k_2(z,y)\, k_1(y,x) = k_2(z,\tilde{y})\, k_1(y,x)\Big|_{y=\tilde{y}}$$

$$\mathrm{Supp}\, k_2 \circ k_1 \subset \{ (z,x) : \exists\, y \text{ such that } (z,y) \in \mathrm{Supp}\, k_2 \text{ and } (y,x) \in \mathrm{Supp}\, k_1 \}$$

## Composition of Canonical Relations:

If $C_1$ is a canonical relation from $T^*Y\setminus 0$ to $T^*X\setminus 0$ and $C_2$ another from $T^*Z\setminus 0$ to $T^*Y\setminus 0$, we define

$$C_1 \circ C_2 = \{ (x,\xi,z,\zeta) \in T^*X\setminus 0 \times T^*Z\setminus 0 ;$$

$$\exists\ (y,\eta) \in T^*Y\setminus 0 \text{ such that}$$

$$(x,\xi,y,\eta) \in C_1 \text{ and } (y,\eta,z,\zeta) \in C_2 \}$$

Let

$$\tilde{D} = (T^*X\setminus 0) \times \mathrm{Diag}\big((T^*Y\setminus 0)\times(T^*Y\setminus 0)\big) \times (T^*Z\setminus 0)$$

Then

$$C_1 \circ C_2 = \Pi\big((C_1 \times C_2) \cap \tilde{D}\big)$$

where

$$\pi : (C_1 \times C_2) \cap \tilde{D} \longrightarrow (T^*X \setminus 0) \times (T^*Z \setminus 0)$$

$$(x, \xi, y, \eta, y, \eta ; z, \zeta) \longmapsto (x, \xi, z, \zeta)$$

Theorem: Let $X$, $Y$ and $Z$ be $C^\infty$ manifolds. Let $A_1 \in I^{m_1}(X \times Y, C_1', \Omega^{1/2}_{X \times Y})$ and $A_2 \in I^{m_2}(Y \times Z, C_2', \Omega^{1/2}_{X \times Y})$ be the kernels of F.I.O ~~For~~ $\mathcal{F}_1$ and $\mathcal{F}_2$. Suppose that

(i) The projection from

$$(\mathrm{Supp}\, A_1 \times \mathrm{Supp}\, A_2) \cap (X \times \mathrm{Diag}(Y \times Y) \times Z)$$

into $X \times Z$ is proper.

(ii) $C_1 \subset T^*X \setminus 0 \times T^*Y \setminus 0$, $C_2 \subset (T^*Y \setminus 0) \times (T^*Z \setminus 0)$ are conic and closed.

(iii) $C_1 \circ C_2 \subset (T^*X \times T^*Z) \setminus 0$ is closed.

(iv) $C_1 \times C_2$ intersects $\tilde{D}$ transversally.

Then $C_1 \circ C_2$ is a homogeneous canonical relation from $T^*Z \setminus 0$ to $T^*X \setminus 0$ and.

$$A_1 \circ A_2 \in I^{m_1+m_2}(X \times Z; (C_1 \circ C_2)', \Omega^{1/2}_{X \times Z}).$$