Pseudo differential Operators

Definition. $A \in \Psi^m(\mathbb{R}^d)$ if there exists $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$ such that

$$A\varphi(x) = \int e^{i\langle x,\xi\rangle} a(x,\xi)\,\hat{\varphi}(\xi)\,d\xi.$$

We can write (locally)

$$A\varphi(x) = \int e^{i\langle x,\xi\rangle} a(x,\xi)\left(\int e^{-i\langle y,\xi\rangle}\varphi(y)\,dy\right)d\xi$$

$$= \int\left(\int e^{i\langle x-y,\xi\rangle} a(x,\xi)\,d\xi\right)\varphi(y)\,dy$$

$$K_A(x,y) = \int e^{i\langle x-y,\xi\rangle} a(x,\xi)\,d\xi$$

is the kernel of A

If we set $x'' = x$; $x - y = x'$, then.

$$I(x', x'') = K_A(x'', x'' - x') = \int e^{i\langle x',\xi\rangle} a(x'',\xi)\,d\xi$$

Hence $A \in \Psi^m(\mathbb{R}^d)$ if and only if. (2)

(*) $$x'^{\alpha'} x''^{\alpha''} D_{x'}^{\beta'} D_{x''}^{\beta''} K_A \in H_{loc}^{\infty, -m-d/2}$$

provided $|\alpha'| \geq \beta'$

$$x' = x - y, \quad x'' = x; \qquad \partial_{x_j} = \partial_{x_j''} + \partial_{x_j'}; \ \partial_{y_j} = -\partial_{x_j'}$$

$$\partial_{x_j} + \partial_{y_j} = \partial_{x_j''}; \qquad \partial_{x_j'} = -\partial_{y_j}$$

So we translate (*) into:

$$(x-y)^{\alpha'} D_y^{\beta'} (D_x + D_y)^{\beta''} K_A \in H_{loc}^{\infty, -m-d/2}$$

$$|\alpha'| \geq \beta'$$

We want to define Conormal distributions to a C^∞ surface of codimension k to be Compatible with this ~~definition~~. Conclusion.

Suppose $\Sigma \subset \mathbb{R}^n$ is a C^∞ surface of codimension k. Let

$\mathcal{V}$ be the space of C^∞ vector fields tangent to Σ.

We say that

$$u \in I^m(\mathbb{R}^n, \Sigma) \text{ if}$$

$$\mathcal{V}^\alpha u \in H^{\infty, -m-n/4}_{loc}(\mathbb{R}^n).$$

In this case, in local coordinates in which

$$\Sigma = \{ x_1 = \dots = x_k = 0 \}$$

$$u = \int e^{i\langle x', \xi' \rangle} a(x'', \xi') \, d\xi'$$

$$a \in S^{m + (n-2k)/4}(\mathbb{R}^{n-k} \times \mathbb{R}^k).$$

This choice makes $A \in \Psi^m(\mathbb{R}^d) \iff$

$$K_A \in I^m(\mathbb{R}^d \times \mathbb{R}^d; \mathrm{Diag}).$$

In this case $n = 2d$; $k = d$. $\underline{\underline{a \in S^m}}$.

C^∞ manifolds, Vector Bundles, Distributions on C^∞ manifolds

I. Topological spaces X is a set; $\mathcal{F}$ is a family of subsets of X such that,

i) $X \in \mathcal{F}$, $\phi \in \mathcal{F}$.

ii) If $U_\alpha \in \mathcal{F}$, $\alpha \in A$ an index set

$$\bigcup_{\alpha \in A} U_\alpha \in \mathcal{F}.$$

iii) If $U_j \in \mathcal{F}$, $\bigcap_{j=1}^{J} U_j \in \mathcal{F}$.

$(X, \mathcal{F})$ is said to be a topological space

$\mathcal{F}$ = open sets of X.

If $\mathcal{B} \subset \mathcal{F}$ is a subfamily of $\mathcal{F}$,

We say that $\mathcal{B}$ is a basis if for all $U \in \mathcal{F}$, $U = \bigcup_{\alpha \in A} B_\alpha$, $B_\alpha \in \mathcal{B}$.

If $\mathcal{B}$ is countable, we say that $(X, \mathcal{F})$ has a countable basis.

We say that a topological space $(X, \mathcal{F})$ is a Hausdorff space if for all $p, q \in X$, there exist $U_p, U_q \in \mathcal{F}$, $p \in U_p$, $q \in U_q$

$$U_p \cap U_q = \phi.$$

<u>Obvious example</u>: $\mathbb{R}^n$. $\mathcal{B} = \{ B(x, r) ; x \in \mathbb{R}^n, r > 0 \}$

$$B(x, r) = \{ y \in \mathbb{R}^n : |y - x| < r \}.$$

$\mathcal{F}$ is the family generated by $\mathcal{B}$.

<u>Countable basis</u>. $\{ B(x_j, r_j) : x_j \in \mathbb{Q}^n ; r_j \in \mathbb{Q}^+ \}$.

<u>X is a C^∞ manifold</u> if.

(i) $(X, \mathcal{F})$ is a Hausdorff space with a countable basis $\mathcal{B}$.

(ii) For each $p \in X$, there exists $U_p \in \mathcal{B}$ (called a neighborhood of p) and a map

$$\psi : U_p \longrightarrow W_p \subset \mathbb{R}^n, \text{ open.}$$

which is continuous and has continuous

inverse.

(iii) A C^∞ structure on X is a family

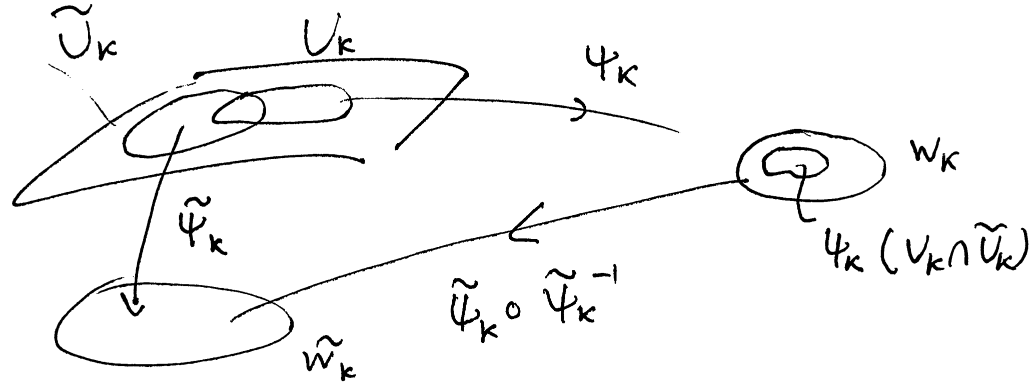
$\tau^{\infty} = \{ (U_k, \psi_k) \}$ $U_k \in \mathcal{B}$ and

$$\psi_K : U_K \longrightarrow W_K \subset \mathbb{R}^n$$

Such that if $(U_K, \psi_K); (\tilde{U}_K, \tilde{\psi}_K) \in \tau^\infty$

$$\tilde{\psi}_K \circ \psi_K^{-1} : \psi_K(U_K \cap \tilde{U}_K) \longrightarrow \tilde{\psi}_K(U_K \cap \tilde{U}_K)$$

is a C^∞ diffeomorphism



(iv) $$X = \bigcup_{U_K \in \tau^\infty} U_K$$

Such a family is called an atlas.

(v) <u>Completion of an atlas</u>; Let $\tilde{\tau}$ be the family of (X_K, φ_K) $X_K \in \mathcal{B}$ such that

$\varphi_k : X_k \longrightarrow V_k \subset \mathbb{R}^n$

is continuous with continuous inverse.

and such that for all $(U_k, \psi_k) \in \mathcal{T}$

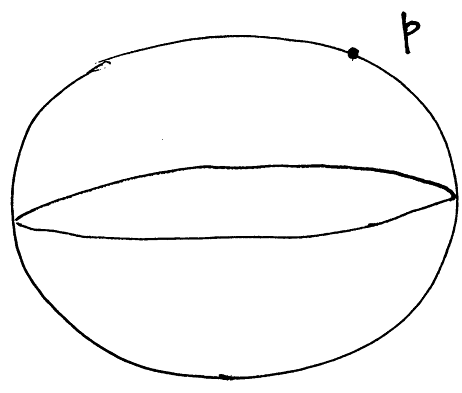
$$\psi_k \circ \varphi_k^{-1} : \varphi_k(X_k \cap U_k) \to \psi_k(X_k \cap U_k)$$

is a C^∞ diffeomorphism.

A C^∞ structure is a complete atlas.

two atlases are equivalent if they define the same C^∞ structure.

<u>Examples</u>:



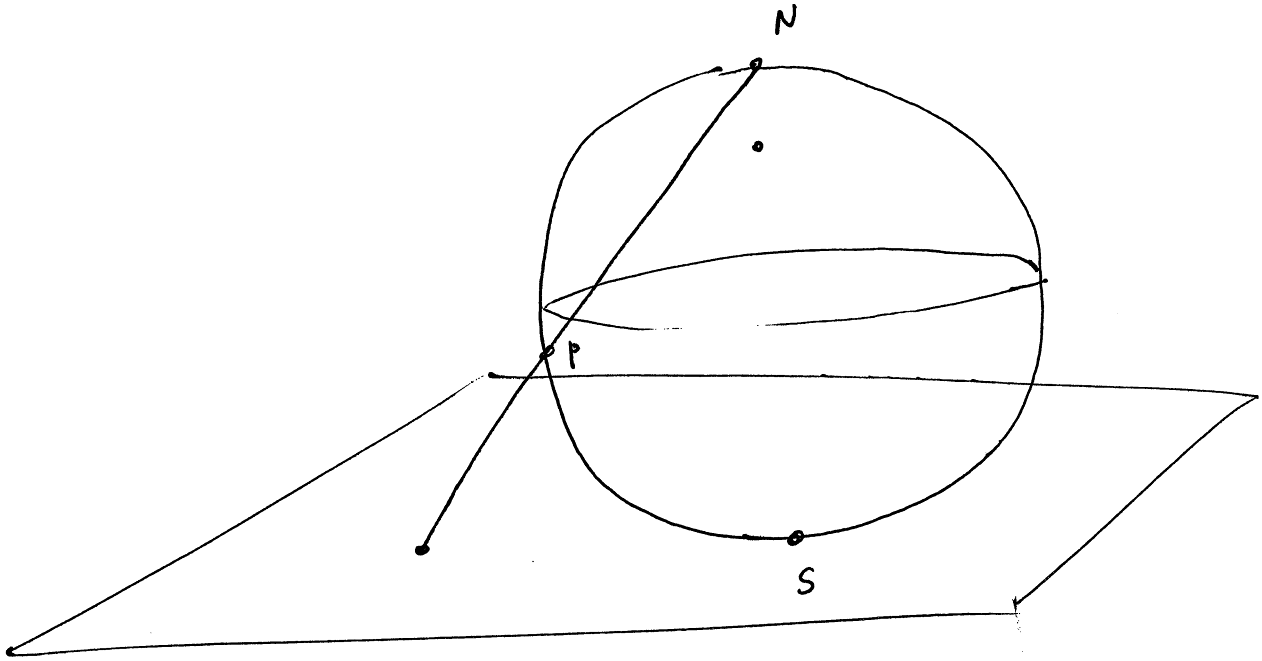
$S^{n-1} = \{ x \in \mathbb{R}^n ; |x| = 1 \}$

<u>S^{n-1}</u>: <u>Atlases</u>: $p \in S^{n-1}$ $U_p = \{ x_j(p) > 0 \}$

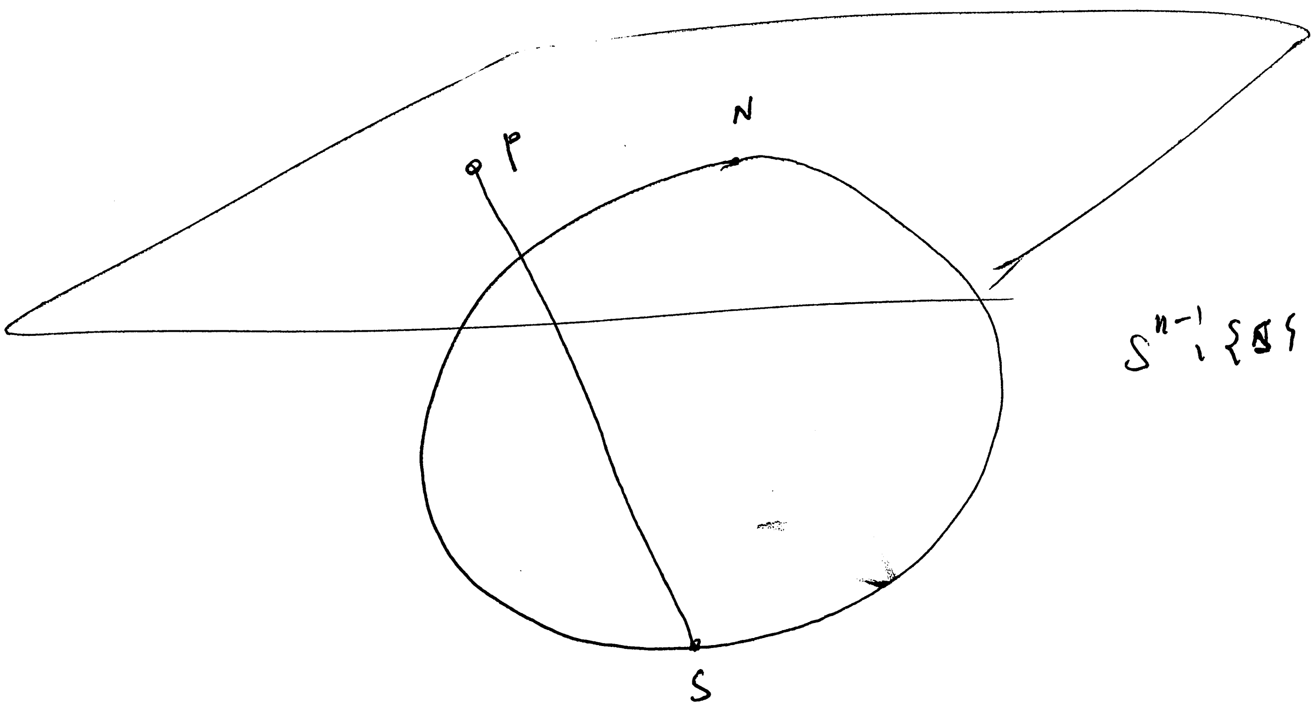
$x' = (x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$,

$$x_j = \sqrt{1 - |x'|^2}$$

U_1, $\mathbb{R}^{n-1}$ $|x'| < 1$,



$S^{n-1} \setminus \{N\}$



$S^{n-1} \setminus \{S\}$

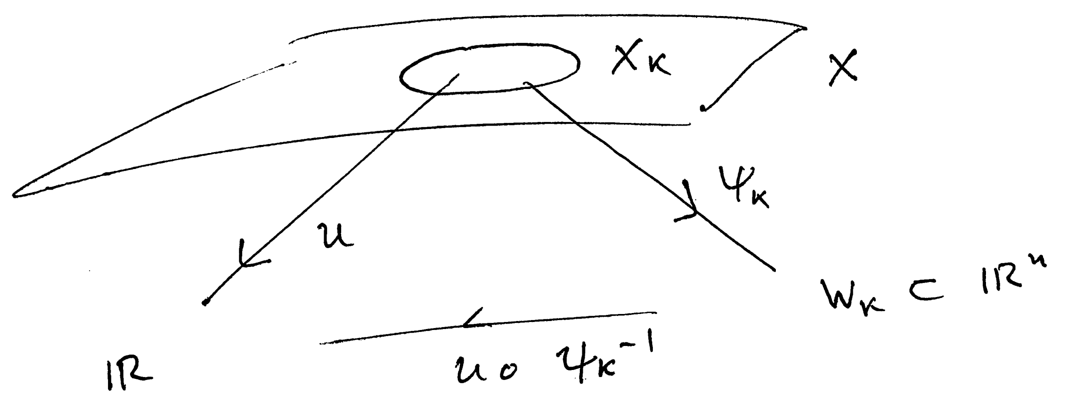
functions on C^∞ manifolds

Let X be a C^∞ manifold and

let $u : X \longrightarrow \mathbb{R}$ be a function.

Def: $u \in C^{m}(X)$ or $u \in L^p_{loc}(X)$, etc, if

for every (X_κ, ψ_κ) in the C^∞ structure of X



$$u_\kappa = u \circ \psi_\kappa^{-1} = (\psi_\kappa^{-1})^* u : W_\kappa \longrightarrow \mathbb{R}$$

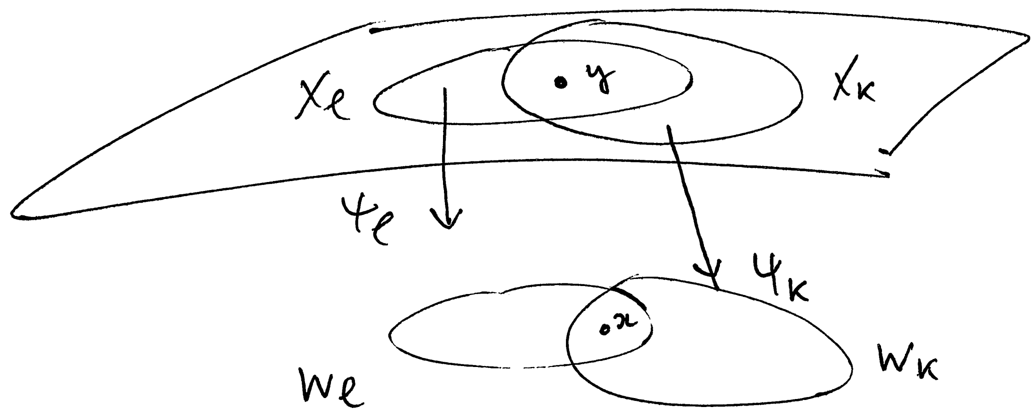
is in $C^{m}(W_\kappa)$ or $L^p(W_\kappa)$

Exercise: Verify that it is enough to require this for an atlas.

If $u \in C^m(X)$, then for every chart

(X_k, ψ_k) $\quad u_k = u \circ \psi_k^{-1} \in C^m(W_k)$

$W_k = \psi_k(X_k)$



$$u_k(x) = u \circ \psi_K^{-1}(x) = u \circ \psi_\ell^{-1}(x) = u_\ell(x)$$

$$u = u_k \circ \psi_k = u_\ell \circ \psi_\ell \quad \text{on } X_\ell \cap X_K.$$

Hence $\quad u_k \circ (\psi_k \circ \psi_\ell^{-1}) = u_\ell.$

in $\psi_\ell(X_k \cap X_\ell)$.

Suppose that to every chart (X_K, ψ_K) of the C^∞ structure of X, we have $u_K \in C^m(W_K)$ $\quad W_K = \psi(X_K)$.

in such a way that.

$$u_\ell = u_k \circ (\psi_k \circ \psi_\ell^{-1}) \text{ in } \psi_\ell(X_k \cap X_\ell)$$

then there exists a unique $u : X \longrightarrow \mathbb{R}$

such that $u_k = u \circ \psi_k^{-1}$.

Let X be a C^∞ manifold.

<u>Definition</u>: Let X be a C^∞ manifold. If for every chart (X_k, ψ_k) in the C^∞ structure of X we are given $u_k \in \mathcal{D}'(W_k)$

$W_k = \psi_k(X_k)$ such that

$$u_\ell = (\psi_k \circ \psi_\ell^{-1})^* u_k.$$

~~where~~

in $\psi_\ell(X_k \cap X_\ell)$.

We shall denote $u_k = u \circ \psi_k^{-1}$

Recall that if $u \in \mathcal{D}'(X)$ and $\psi : Y \longrightarrow X$ is a C^∞ diffeomorphism: $\psi^* u \in \mathcal{D}'(Y)$

and $\forall\ \varphi \in C_0^\infty(Y)$

$$\langle \psi^* u, \varphi \rangle = \langle u, \varphi \circ \psi^{-1} \rangle.$$

Exercise: Let X be a C^∞ manifold and let $\mathcal{L}$ be an atlas for X. Suppose that for every $(X_\kappa, \psi_\kappa) \in \mathcal{L}$ there exists $u_\kappa \in \mathcal{D}'(W_\kappa)$; $W_\kappa = \psi_\kappa(X_\kappa)$, then there exists a unique $u \in \mathcal{D}'(X)$ such that

$$u \circ \psi_\kappa^{-1} = u_\kappa .$$