MA 693 : Lecture 7

Vector Bundles :

X is a C^∞ manifold.

Definition: A C^∞ real vector bundle over X of dimension N is a C^∞ manifold V with.

(i) A C^∞ map $\pi : V \longrightarrow X$ called the projection

(ii) For each $x \in X$, $V_x = \pi^{-1}(x)$ is a vector space of dimension N

(iii) For each $x \in X$, there exists a neighborhood $Y \ni x$ ~~patch~~ ~~that~~ and a map

$$\psi : Y \times \mathbb{R}^N \longrightarrow \pi^{-1}(Y)$$

Such that ψ is a C^∞ diffeomorphism and.

(a) $\pi \circ \psi(x, v) = x \quad \in v \in \mathbb{R}^N$

(b) The map $v \mapsto \psi(x, v)$

$$\mathbb{R}^N \longmapsto \pi^{-1}(x) = V_x$$

$$v \longmapsto \psi(x, v)$$

is an isomorphism (invertible linear transformation).

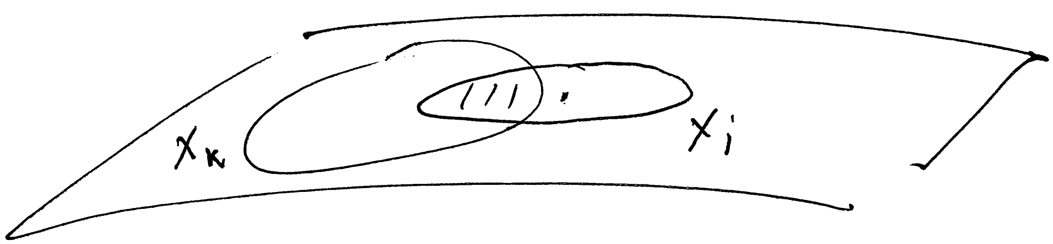
Let X be a C^∞ manifold and let V be a real vector bundle over X of fiber dimension N.

Choose an open covering $\{X_j\}_{j \in J}$ of X such that for each X_j, $j \in J$,

$$\exists \quad \psi_j : X_j \times \mathbb{R}^N \longrightarrow \pi^{-1}(X_j)$$

as above. If $X_j \cap X_k \neq \emptyset$

$$\psi_k : X_k \times \mathbb{R}^N \longrightarrow \pi^{-1}(X_k).$$



$$\psi_j : (X_j \cap X_k) \times \mathbb{R}^N \longrightarrow \pi^{-1}(X_j \cap X_k)$$

$$\psi_k : (X_j \cap X_k) \times \mathbb{R}^N \longrightarrow \pi^{-1}(X_j \cap X_k)$$

$$\psi_k^{-1} \circ \psi_j : (X_j \cap X_k) \times \mathbb{R}^N \longrightarrow (X_j \cap X_k) \times \mathbb{R}^N$$

$$\psi_k^{-1} \circ \psi_j (x, v) = (x, g_{kj}(x) v)$$

$g_{kj}(x)$ is an invertible $N \times N$ matrix

$$g_{kj} : X_j \cap X_k \longrightarrow G(L, N)$$

We have the following compatibility conditions.

(a) $g_{jk} \, g_{kj} = Id$ on $X_j \cap X_k$

(b) $g_{jk} \, g_{k\ell} \, g_{\ell j} = Id$ on $X_j \cap X_k \cap X_\ell$

A system $\{ g_{jk}(x) ; X_j \subset X ; j \in J \} =$

is called a system of transition matrices.

If we are given such a system

We say that

$$(j, x, v) \equiv (k, y, w)$$

if $x = y$ and $w = g_{kj}(x) v$.

This is an equivalence relation

(A) $(j, x, v) \equiv (j, x, v)$

$$g_{jj}(x) = Id$$

(B) If $(j, x, v) \equiv (k, y, w)$

then $(k, y, w) = (j, x, v)$

since $g_{jk}(x) = g_{kj}^{-1}(x)$

(C) If $(j, x, v) \equiv (k, y, w)$

and $(k, y, w) \equiv (\ell, z, u)$

then $x = y = z$ $\quad w = g_{kj}(x)\, v$

$$u = g_{\ell k}(x)\, w$$

then $u = g_{\ell k}(x)\, g_{kj}(x)\, v$.

But $g_{j\ell}(x)\, g_{\ell k}(x)\, g_{kj}(x) = Id$

$$g_{\ell k}(x)\, g_{kj}(x) = g_{j\ell}^{-1}(x) = g_{\ell j}(x)$$

So $u = g_{\ell j}(x)\, v$.

Proposition: Let

$$V' = \{ (j, x, v) \,;\, j \in J,\ x \in X_j,\ v \in \mathbb{R}^N \}$$

then $V'/\sim$ is a C^∞ vector bundle

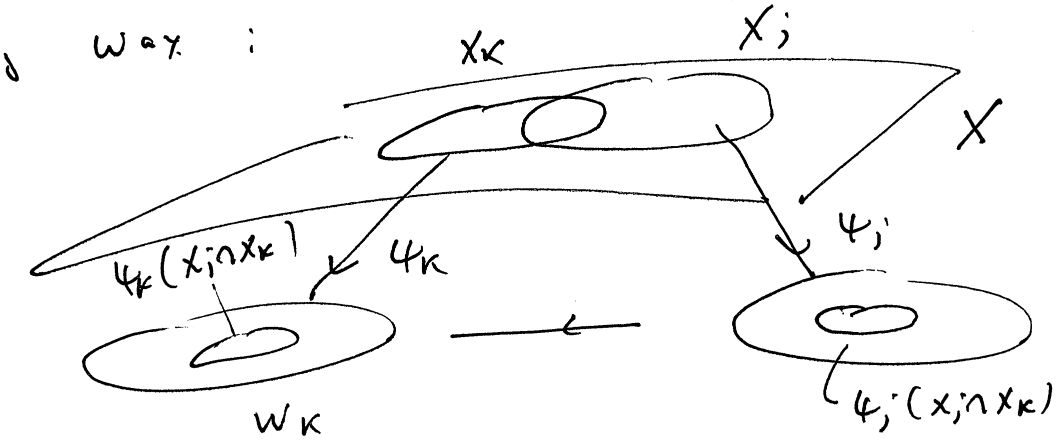
and if V is a vector bundle over X

and $g_{ij}(x)$ are the transition matrices

for V, then $V \simeq V'/\sim$.

Example: α-density bundles are line bundles

over a C^∞ manifold X defined in the following way: Let $\{(X_K, \psi_K)\}$ be an atlas of X.



$$\psi_j \circ \psi_K^{-1} : \psi_K(X_j \cap X_K) \to \psi_j(X_K \cap X_j)$$

is C^∞ diffeomorphism

$$\left|\det\left(\psi_j \circ \psi_k^{-1}\right)'\right| \neq 0. \quad \text{defined on } \psi_k(X_j \cap X_k).$$

Transition functions (matrices)

$$X_j \times \mathbb{R} \xrightarrow{\psi_j} \pi^{-1}(X_j)$$

$$X_j \cap X_k \times \mathbb{R} \longrightarrow X_j \cap X_k \times \mathbb{R}$$

$$(x, t) \longmapsto (x, g_{jk}(x)\, t)$$

Define $\quad g_{jk}(x) = \left|\det\left(\psi_j \circ \psi_k^{-1}\right)'\right|^{\alpha} \circ \psi_k^{-1}$

We ~~define~~ denote this bundle by $\Omega^{\alpha}(X)$.

<u>Sections of Vector Bundles</u>: If $Y \subset X$ is an open subset, a map

$$u : Y \to V$$

such that $\pi(u(y)) = y$ is called a section of the vector bundle V.

Since V is a C^∞ manifold,

$C^k(Y; V)$ is well defined.

Let (X_j, ψ_j) be a covering of $X = \bigcup X_j$

$\psi_j : X_j \times \mathbb{R}^N \longrightarrow \pi^{-1}(X_j)$, then

if $u : Y \longrightarrow V$ is a section.

~~use~~

$$X_j \cap Y \xrightarrow{u} V|_{X_j} \xrightarrow{\psi_j^{-1}} X_j \times \mathbb{R}^N.$$

$$u_j = \psi_j^{-1} \circ u : \quad X_j \cap Y \longrightarrow X_j \times \mathbb{R}^N$$

$$x \longmapsto (x ; \psi_j^{-1} \circ u(x))$$

$$u_j \in C^k(Y \cap X_j ; \mathbb{R}^N)$$

$$\psi_j \circ u_j = \psi_k \circ u_k \quad \text{on} \quad Y \cap X_j \cap X_k$$

So $u_j = \underbrace{\psi_j^{-1} \circ \psi_k}_{} \circ u_k$

$(**)$ $u_j = g_{jk} u_k$. in $Y \cap X_j \cap X_k$.

<u>Exercise</u>: If (X_j, ψ_j) are as above and $u_j \in C^m(Y \cap X_j ; \mathbb{R}^N)$ satisfy $(**)$

then there is a unique $u \in C^m(Y, V)$.

We can define the space of distribution

Sections $u \in \mathcal{D}'(Y; V)$ as the space of all systems $u_j \in \mathcal{D}'(Y \cap X_j, \mathbb{R}^N)$ satisfying $(**)$.

<u>Density - Valued distributions</u> : Let $\alpha \in \mathbb{R}$ and let $\Omega^\alpha(X)$ be the α-density bundle. $u \in \mathcal{D}'(X; \Omega^\alpha(X))$ are the α-density valued distributions.

<u>Exercise</u> : Let $C_0^\infty(X, \Omega^{1-\alpha}(X))$ be the space of compactly supported C^∞ sections of $\Omega^{1-\alpha}(X)$. Then

$\mathcal{D}'(X, \Omega^\alpha(X))$ is the dual to

$C_0^\infty(X; \Omega^{1-\alpha}(X))$.