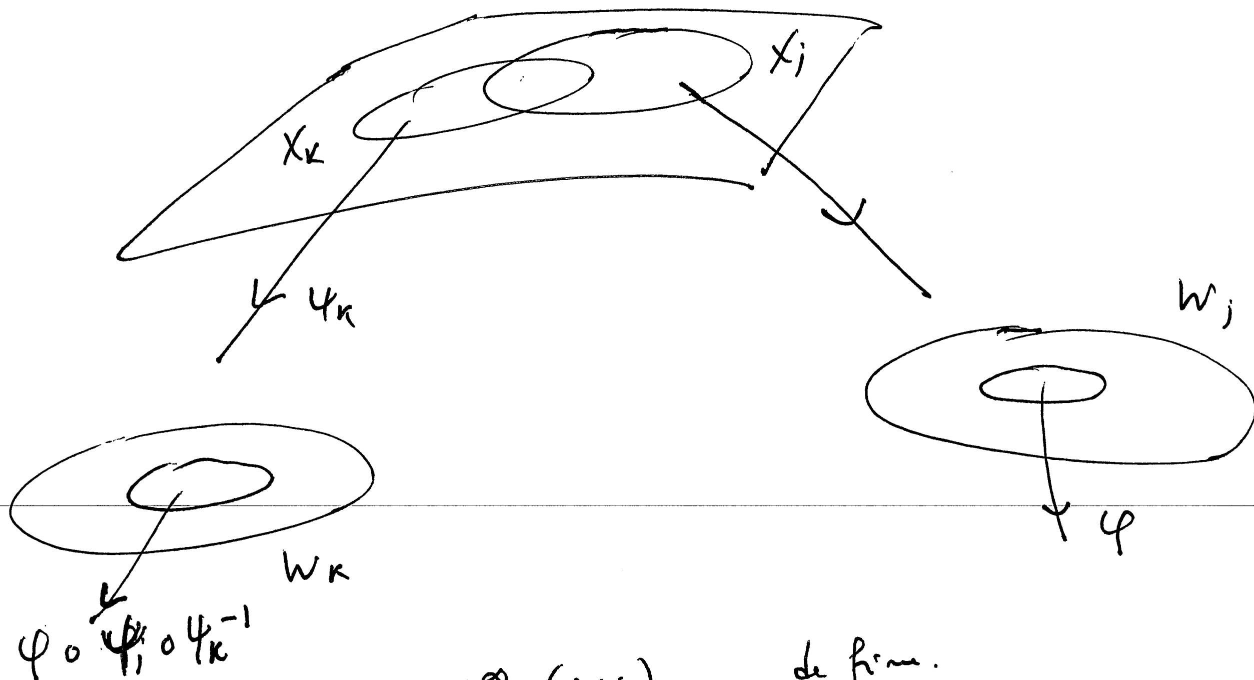
Lecture 8: Vector Bundles (Continuation)

Density-Valued Distributions

The case $\alpha = 0$: Find the dual to $C_0^\infty(X)$.

Let $u : C_0^\infty(X) \longrightarrow \mathbb{C}$ be a continuous linear functional



Let $\varphi \in C_0^\infty(W_j)$, define.

$$\langle u_j, \varphi \rangle = \langle u, \varphi \circ \psi_j \rangle$$

$\varphi \circ \psi_j(x) = 0$ outside X_j

If $\varphi \in C_0^\infty(\psi_j(X_j \cap X_k))$

$$\langle u_k ; \varphi \circ \psi_j \circ \psi_k^{-1} \rangle = \langle u, \varphi \circ \psi_j \rangle$$

$$\langle u_j , \varphi \rangle = \langle u, \varphi \circ \psi_j \rangle$$

Hence $\langle u_j , \varphi \rangle = \langle u_k , \varphi \circ F \rangle \qquad F = \psi_j \circ \psi_k^{-1}$

$F(x) = y$

But $\langle u_k , \varphi \circ F \rangle = \int u_k(x) \, \varphi \circ F(x) \, dx$

$$= \int u_k (F^{-1}(y)) \, \varphi(y) \, |(F^{-1})'(y)| \, dy$$

$$= \langle (F^{-1})^* u_k \, |(F^{-1})'| , \varphi \rangle$$

Hence (A) $u_j = G^* u_k \, | \det G' | . \quad G = \psi_k \circ \psi_j^{-1}$

Conversely, suppose we are given distributions u_j on W_j satisfying (A) for all (X_j, ψ_j) in an atlas of X. Choose a partition of unity $\chi_j \in C_0^\infty(X_j)$ and set

$$\langle u, \varphi \rangle = \sum_j \langle u_j ; (\chi_j \varphi) \circ \psi_j^{-1} \rangle .$$

$\varphi \in C_0^\infty(X)$.

The Tangent Bundle

Model Case: $X = \mathbb{R}^N$; $V = \mathbb{R}^N \times \mathbb{R}^N$

$$v = (a_1, a_2, \dots, a_N) = \sum_{j=1}^{N} a_j e_j \qquad e_j = (0, \dots, 1, 0 \dots, 0)$$

· Consider the map : $\psi : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow V$

$$\psi(x, v) = \left(x ; \sum_{j=1}^{N} a_j(x) \frac{\partial}{\partial x_j}\right)$$

Transition Matrices. Let $w = (b_1(x), \dots, b_N(x))$

$$\tilde{\psi}(y, w) = \left(y ; \sum_{j=1}^{N} b_j(y) \frac{\partial}{\partial y_j}\right)$$

Suppose that $y = F(x)$ and $y_0 = F(x_0) = x_0$

· $y = (F_1(x), F_2(x), \dots, F_N(x))$.

$x = (G_1(y), G_2(y), \dots, G_N(y)) = (x_1, \dots, x_N)$

$$\frac{\partial}{\partial y_j} = \sum_K \frac{\partial x_K}{\partial y_j} \frac{\partial}{\partial x_K} = \sum_K \frac{\partial G_K}{\partial y_j}(y) \cdot \frac{\partial}{\partial x_K}$$

$$= \sum_K \frac{\partial G_K}{\partial y_j}(F(x)) \frac{\partial}{\partial x_K}.$$

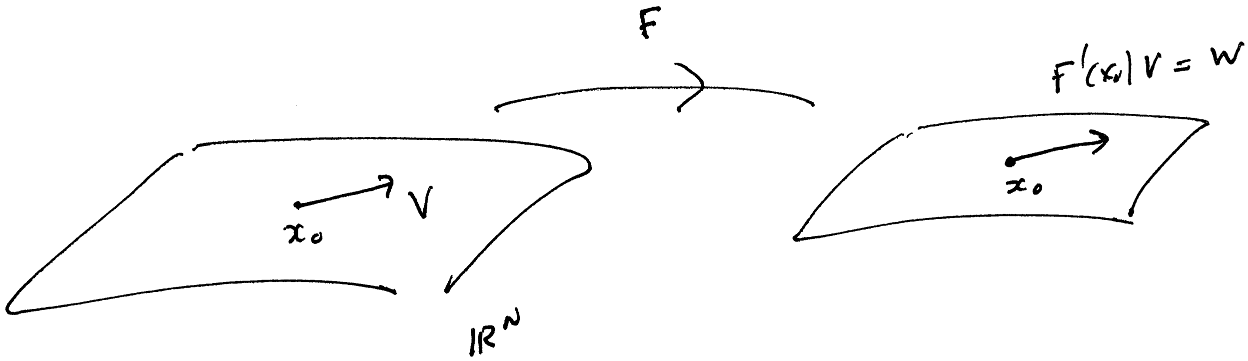
Hence

$$\tilde{\psi}(x_0, w) = \left(x_0; \sum_{k,j=1}^{N} b_j(x_0) \frac{\partial G_k}{\partial y_j}(x_0) \frac{\partial}{\partial x_k}\right)$$

$$a_k = \sum_j \frac{\partial G_k}{\partial y_j}(x_0) \cdot b_j(x_0) = \left[(F^{-1})'(x_0)\, w\right]_k$$

$$v = (F^{-1})'(x_0)\, w.$$

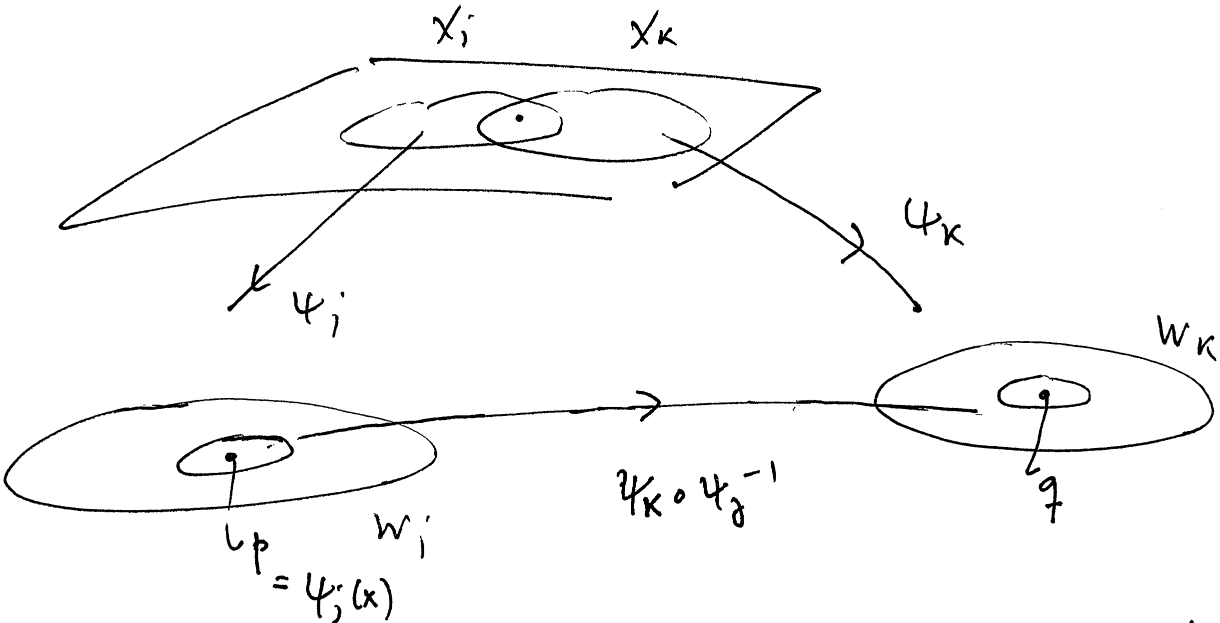
$$w = (F)'(x_0)\, v.$$



The transition matrix $= F'(x_0)$.

The case of C^∞ manifolds:

Let X be a C^∞ manifold and let (X_i, ψ_i) be an atlas for X.



Define the transition matrices on $X_j \cap X_K$.

$$G_{jK} = (\psi_K \circ \psi_j^{-1})' (\psi_j(x)).$$

One can check that $G_{jj} = Id$.

$$G_{jK} G_{Kj} = Id$$

$$G_{jK} G_{Kl} G_{lj} = Id.$$

Locally we are identifying the tangent bundle of X with the tangent bundle of X_j

We define this bundle as TX.

Sections of this bundle are locally of the form.

$$v(x) = \sum_{j=1}^{m} a_j(x) \frac{\partial}{\partial x_j}.$$

<u>The dual of a vector bundle</u>: Let X be a C^∞ manifold and let $\begin{matrix} V \\ \downarrow \pi \\ X \end{matrix}$ be a C^∞ vector bundle over X

of fiber dimension N.

Let $x \in X$ and $Y \ni x$ be an open neighborhood of x. and let

$$\begin{aligned} \psi : Y \times \mathbb{R}^N &\longrightarrow \pi^{-1}(Y) \\ (x, v) &\longmapsto \psi(x, v). \end{aligned}$$

For each x, $\psi_x = \psi(x, \cdot) : \mathbb{R}^N \longrightarrow V_x = \pi^{-1}(x)$ is an invertible linear transformation.

This induces a map $\psi_x^* : V_x^* \longrightarrow \mathbb{R}^N$

and hence $(\psi_x^*)^{-1} : \mathbb{R}^N \longrightarrow V_x^*$

and is an invertible linear transformation

We want to define the bundle V^* with maps

$$\tilde{\Psi} : Y \times \mathbb{R}^N \longrightarrow \pi^{-1}(V^*)$$

$$(x, v) \longmapsto \tilde{\Psi}(x, v)$$

Fixed x; $\tilde{\Psi}_x(v) = \tilde{\Psi}(x, v) = (\Psi_x^*)^{-1} v$.

If the transition matrices for V are.

$$\Psi_k^{-1} \circ \Psi_j (x, v) = (x; g_{kj}(x) v)$$

the transition matrices for V^* are

$$(g_{kj}^{-1})^* = g_{jk}^* .$$