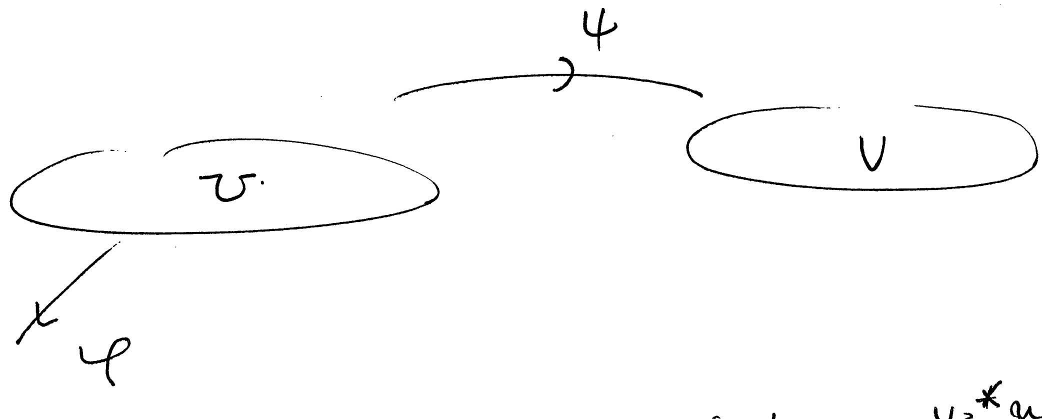
Density Valued Distributions. (Addendum) Lesson 3 ①

Recall that if $U, V \subset \mathbb{R}^n$ are open subsets and

$$\psi : U \longrightarrow V$$

is a C^∞ diffeomorphism.

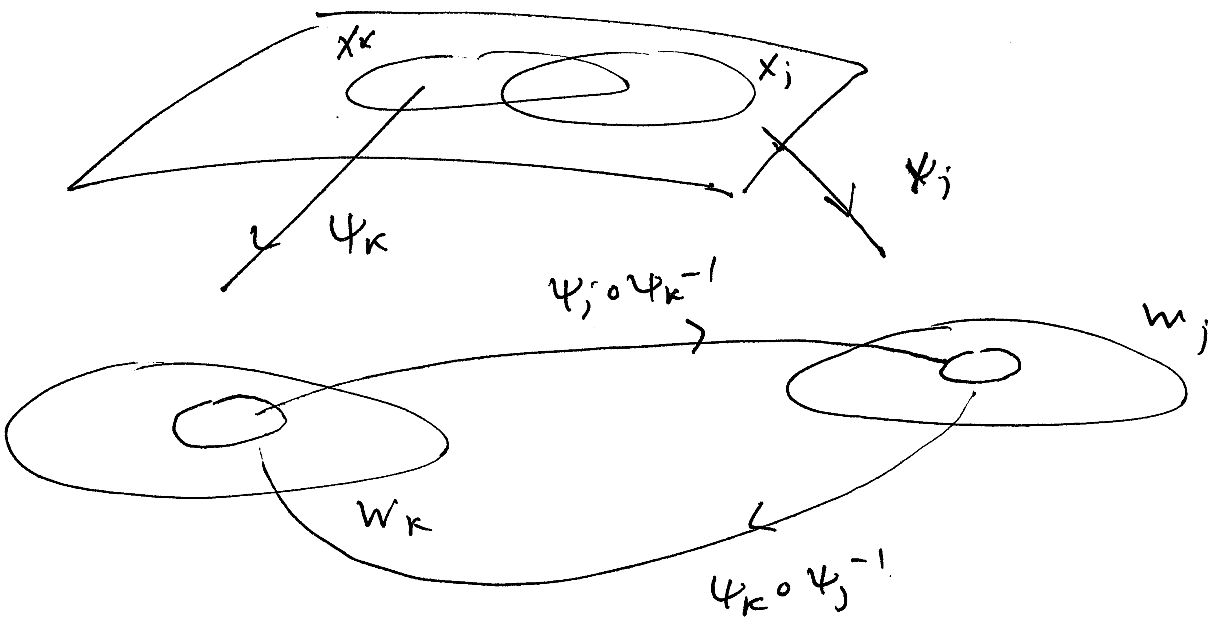


and if $u \in \mathcal{D}'(V)$, the pull-back $\psi^* u \in \mathcal{D}'(U)$ is defined to be.

$$\text{(PBD)} \quad \langle \psi^* u, \varphi \rangle = \langle u, \varphi \circ \psi^{-1} \rangle$$

$$\forall \varphi \in C_0^\infty(U).$$

Distributions on Manifolds Let X be a C^∞ manifold and $\{(X_i, \psi_i)\}$ the maximal atlas for X.



We are given a family of distributions $u_j \in \mathcal{D}'(W_j)$

$W_j = \psi_j(X_j)$ such that for any

$$\varphi \in C_0^\infty(\psi_j(X_j \cap X_k))$$

(PB1) $\langle u_j, \varphi \rangle = \langle u_k, \varphi \circ (\psi_j \circ \psi_k^{-1}) \rangle$

or according to (PB)

$$u_j = (\psi_k \circ \psi_j^{-1})^* u_k \quad \text{on} \quad \psi_j(X_j \cap X_k)$$

The system $u = \{ u_j \in \mathcal{D}'(W_j) \}$ is defined to be a distribution on X and we denote $u \in \mathcal{D}'(X)$.

One can use $u_j \in \mathcal{D}'(W_j)$ to define

$$\tilde{u}_j \in \mathcal{D}'(X_j) \quad \text{by} \quad \tilde{u}_j = \psi_j^* u_j$$

$$\langle \tilde{u}_j, \varphi \rangle = \langle u_j, \varphi \circ \psi_j^{-1} \rangle \qquad \varphi \in C_0^\infty(X_j)$$

But (PB1) implies that

$$\langle \tilde{u}_j, \varphi \rangle = \langle \tilde{u}_k, \varphi \rangle \qquad \varphi \in C_0^\infty(X_j \cap X_k)$$

So if $\varphi \in C_0^\infty(X)$ one can define

$$\langle u, \varphi \rangle = \sum_{j=1}^{N} \langle u_j, \chi_j \varphi \rangle$$

$$\chi_j \in C_0^\infty(X_j) \qquad \sum_{j=1}^{\infty} \chi_j = 1$$

<u>Density Bundles</u>: Recall that if

$$\begin{array}{c} V \\ \downarrow \pi \\ X \end{array}$$

is a C^∞ vector bundle of fiber dimension N.

for every $x \in X$, there exists a neighborhood Y, $x \in Y$ and a map

$$\varphi: Y \times \mathbb{R}^N \longrightarrow \pi^{-1}(Y).$$

Such that $\varphi(y,\cdot) : \mathbb{R}^N \to V_y = \pi^{-1}(y)$

is an invertible linear transformation.

If $\varphi_j : Y_j \times \mathbb{R}^N \to \pi^{-1}(Y_j)$

$\varphi_k : Y_k \times \mathbb{R}^N \to \pi^{-1}(Y_k)$

are two such transformations. then

$$g_{jk} = \varphi_j^{-1} \circ \varphi_k : (Y_j \cap Y_k) \times \mathbb{R}^N \to (Y_j \cap Y_k) \times \mathbb{R}^N.$$

are the so-called transition matrices.

As we have seen before, $\{g_{jk}, Y_j\}$ determine V.

<u>We define the line bundle</u> $\Omega^\alpha(X)$, $\alpha \in \mathbb{R}$

to be the bundle of fiber dimension 1.

given by the following ~~dense~~ transition matrices.

$$g_{jk} = \left| \det(\psi_j \circ \psi_k^{-1})' \right|^\alpha \circ \psi_j$$

in $X_j \cap X_k$

$\{(X_j, \psi_j)\}$ an atlas of X.

Let $\tilde{u}_j \in \mathcal{D}'(X_j)$ be such that

$$\langle \tilde{u}_j , \varphi \rangle = \langle \tilde{u}_k , \varphi \rangle \qquad \varphi \in C_0^\infty(X_j \cap X_k)$$

$$\tilde{u}_j = \psi_j^* u_j \quad , \quad u_j \in \mathcal{D}'(W_j) \qquad W_j = \psi_j(X_j)$$

We define the distributional valued <u>densities</u>.

$u \in \mathcal{D}'(X; \Omega^\alpha(X))$ such that

$$u = \tilde{u}_j \, \left| \det(\psi_j \circ \psi_k^{-1}) \right|^\alpha \circ \psi_j \qquad \text{on} \quad X_j \cap X_k.$$