MA 693 Introduction To F. I. O

I) A short Review of Distributions

Test Functions. $X \subset \mathbb{R}^n$ open.

$\varphi \in C_0^\infty(X) = C_c^\infty(X)$ if. $\varphi \in C^\infty(X)$ and

$\operatorname{supp} \varphi = \overline{\{x \in X : \varphi(x) \neq 0\}} \subset\subset X$. (is a compact subset of X)

Convergence: $\varphi_j \to \varphi$ in $C_c^\infty(X)$ if

(1) $\sup_{x \in X} |D^\alpha \varphi_j - D^\alpha \varphi| \to 0$ as $j \to \infty$

$D^\alpha \varphi_j = D_1^{\alpha_1}, D_2^{\alpha_2} \dots D_n^{\alpha_n}$ $\alpha = (\alpha_1, \dots, \alpha_n)$

(2) $\exists K \subset\subset X$, $\operatorname{supp} \varphi_j \subset K$.

Distributions.. $u \in \mathcal{D}'(X) = C^{-\infty}(X)$ if

$u : C_0^\infty(X) \longrightarrow \mathbb{C}$ is linear.

$\lim_{j \to \infty} \langle u, \varphi_j \rangle = \langle u, \varphi \rangle$

provided $\varphi_j \to \varphi$ in $C_0^\infty(X)$.

Examples: (1) $\langle \delta_0, \varphi \rangle = \varphi(0)$

(2) $\Sigma \subset \mathbb{R}^n$ a hypersurface

$$\langle \delta_\Sigma, \varphi \rangle = \int_\Sigma (\varphi|_\Sigma)\, d\sigma. \quad \text{(Surface Integral)}$$

Example: $\omega \in S^{n-1}$, $s \in \mathbb{R}$

$$\Sigma_{\omega,s} = \{ x \in \mathbb{R}^n : \langle x, \omega \rangle = s \}$$

$$\langle \delta_{\Sigma_{\omega,s}}, \varphi \rangle = \int_{\langle x,\omega\rangle = s} \varphi(x)\, d\sigma.$$

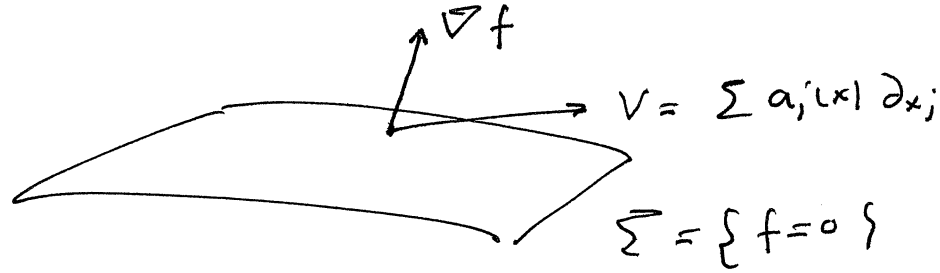
Vector Fields: V is a vector field on $\mathbb{R}^n$.

$$V(x) = \sum_{j=1}^n a_j(x)\, \partial_{x_j} \qquad a_j \in C^\infty(\mathbb{R}^n)$$

$$V\varphi = \sum a_j(x)\, \partial_{x_j} \varphi = \langle (a_1(x), \dots, a_n(x)) ; \nabla \varphi(x) \rangle$$

$\Sigma \subset \mathbb{R}^n$ is a C^∞ hypersurface if $\exists f$ locally.

$$\Sigma = \{ x : f(x) = 0 \} \quad df(x) \neq 0 \text{ on } \Sigma.$$



V is tangent to Σ if $Vf = 0$ on Σ.

__Exercise:__ Show that if Σ is a C^∞ hypersurface and V is a C^∞ vector field, then V is tangent to Σ if and only if for every

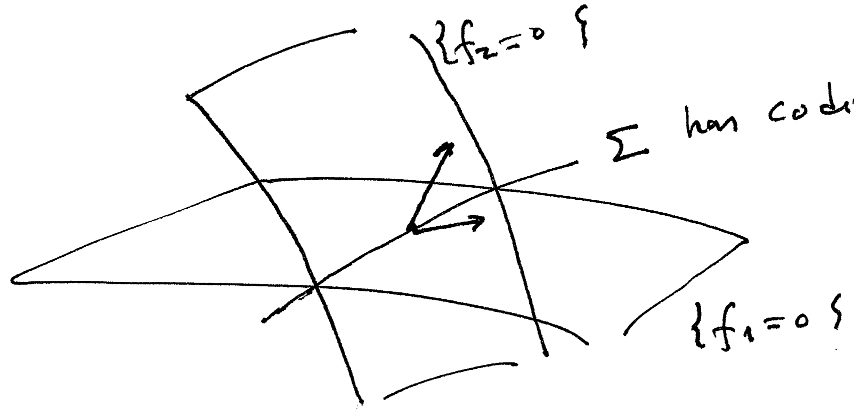
$$\varphi \in C^\infty(\mathbb{R}^n) \qquad \varphi = 0 \text{ on } \Sigma$$

$$V\varphi = 0 \text{ on } \Sigma.$$

__Codimension k surfaces__ : Σ is a C^∞ codimension k surface if for every $p \in \Sigma$ there exists a neighborhood $U \ni p$ such that

$$\Sigma \cap U = \{ f_1(x) = f_2(x) = \dots = f_k(x) = 0 \}$$

$f_j \in C^\infty(U)$ and $\nabla f_1(x), \dots, \nabla f_k(x)$ are linearly independent.



Definition:

Let $V = \sum a_j(x)\, \partial_{x_j}$ be a vector field. ~~then~~

and Σ be a codimension k hypersurface.

Then V is tangent to Σ if

$$\Sigma = \{ f_1 = f_2 = \dots = f_k = 0 \}, \quad df_j \text{ lin. ind } j = 1, 2, \dots, k.$$

$$V f_j = 0 \text{ on } \Sigma.$$

Exercise: Show that V is tangent to Σ

if $\forall\ \varphi \in C^\infty(\mathbb{R}^n)$; $\varphi = 0$ on Σ

$$V\varphi = 0 \text{ on } \Sigma.$$

Tempered Distributions and Fourier transform

<u>Test Functions</u>: $\mathcal{S}(\mathbb{R}^n)$ consists of the space of C^∞ functions φ such that

$$\sup_{x \in \mathbb{R}^n} (1+|x|^2)^N |\partial^\alpha \varphi(x)| \le C_{N,\alpha}(\varphi)$$

$$N \in \mathbb{N}, \quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{N}^m.$$

<u>Convergence</u>: $\varphi_j \in \mathcal{S}(\mathbb{R}^n)$, $\varphi_j \to \varphi$

if

$$C_{N,\alpha}(\varphi_j - \varphi) \to 0 \text{ as } j \to \infty.$$

<u>Tempered Distributions</u>: $u \in \mathcal{S}'(\mathbb{R}^n)$ if

$$u : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}$$

linear and if $\varphi_j \to \varphi$ in $\mathcal{S}(\mathbb{R}^n)$

$$\lim_{j \to \infty} \langle u, \varphi_j \rangle = \langle u, \varphi \rangle.$$

Fourier Transform: If $u \in \mathcal{S}(\mathbb{R}^n)$

$$\hat{u}(\xi) = \int e^{-i\langle x, \xi\rangle}\, u(x)\, dx.$$

Inverse Fourier transform:

$$u(x) = \frac{1}{2\pi}\int e^{i\langle x, \xi\rangle}\, \hat{u}(\xi)\, d\xi. \qquad u \in \mathcal{S}(\mathbb{R}^n)$$

If $u \in \mathcal{S}(\mathbb{R}^n)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\langle \hat{u}(\xi), \varphi(\xi)\rangle = \int \hat{u}(\xi)\, \varphi(\xi)\, d\xi.$$

$$= \int \left(\int e^{-i\langle x, \xi\rangle}\, u(x)\, dx \right) \varphi(\xi)\, d\xi$$

$$= \int \left(\int e^{-i\langle x, \xi\rangle}\, \varphi(\xi)\, d\xi \right) u(x)\, dx$$

$$= \langle u(x) ; \hat{\varphi}(x) \rangle.$$

Definition: $u \in \mathcal{S}'(\mathbb{R}^n)$; $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\langle \hat{u}, \varphi \rangle = \langle u, \hat{\varphi} \rangle$$

Example: $u = \delta_0$;

$$\langle \hat{u}, \varphi \rangle = \langle \hat{\delta_0}, \varphi \rangle = \langle \delta_0, \hat{\varphi} \cdot \rangle = \hat{\varphi}(0)$$

$$= \int \varphi(x)\, dx$$

$$\langle \hat{u}, \varphi \rangle = \langle 1, \varphi \rangle \qquad \hat{u} = 1.$$

Exercise: Compute $\widehat{H}(\xi)$ $\quad H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$

Inverse Fourier transform. If $u \in \mathcal{S}'(\mathbb{R}^n)$

$$\langle \mathcal{F}^{-1} u, \varphi \rangle = \langle u, \mathcal{F}^{-1} \varphi \rangle.$$

- L^2 Space: $f \in L^2(\mathbb{R}^n) \iff \int |f(x)|^2 dx < \infty$

Lebesgue integral.

Exercise: Show that if $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$

$$\int \varphi(x) \overline{\psi(x)}\, dx = \frac{1}{(2\pi)^n} \int \hat{\varphi}(\xi) \overline{\hat{\psi}(\xi)}\, d\xi$$

Plancherel's thm

So $\mathcal{F} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$

<u>Sobolev Spaces</u>: Let $s \in \mathbb{R}$, we say that

$u \in H^s(\mathbb{R}^n)$ if

(1) $u \in \mathcal{S}'(\mathbb{R}^n)$; $\hat{u}(\xi)$ is measurable and

(2) $$\int (1+|\xi|^2)^{s/2} |\hat{u}(\xi)|^2 d\xi = \|u\|_{H^s}^2 < \infty.$$

One can see that

$$H^{s_1}(\mathbb{R}^n) \subset H^{s_2}(\mathbb{R}^n) \quad s_2 \leq s_1$$

$u \in H^s_{loc}(\mathbb{R}^n)$ if $\forall \varphi \in C_0^\infty(\mathbb{R}^n)$

$\varphi u \in H^s(\mathbb{R}^n)$.

Conormal Distributions:

Let $\Sigma \subset \mathbb{R}^n$ be a C^∞ codimension k surface. We say that $u \in C^{-\infty}(\mathbb{R}^n)$ is a conormal distribution with respect to Σ if $\exists\, s \in \mathbb{R}$

$$V_1 . V_2 V_3 \dots V_N u \in H^s_{loc}(\mathbb{R}^n)$$

V_j C^∞ vector field tangent to Σ.

$N \in \mathbb{N}$. We denote this class by $I^s(\Sigma)$.

Finite regularity: $N \leq N_0$.

Example: $$H(x_1) = \begin{cases} 1, & x_1 > 0 \\ 0, & x_1 < 0 \end{cases} \qquad \Sigma = \{x_1 = 0\}$$

Vector fields tangent to Σ:

$V x_1 = 0$ if $x_1 = 0$.

$$V = \sum_{j=1}^{n} a_j(x)\, \partial_{x_j} \; ; \qquad V x_1 = a_1(x) = 0 \text{ if } x_1 = 0$$

Then

$$a_1(x) = x_1 \tilde{a}_1(x).$$ and

therefore

$$V := x_1 \tilde{a}_1(x)\, \partial_{x_1} + \sum_{j=2}^{n} a_j(x)\, \partial_{x_j}$$

$$V\, H(x_1) = 0$$

Local coordinates: Let $\Sigma \subset \mathbb{R}^n$ be a C^∞ surface of codimension k. Then for every point $p \in \Sigma$, there exists a neighborhood U of p and local coordinates $\{y_1, y_2, \dots, y_n\}$ on U such that

$$U \cap \Sigma = \{ y_1 = y_2 = \dots = y_k = 0 \}$$

Case $k=1$: $\Sigma \cap U = \{f = 0\}$, $df = 0$ on $\Sigma \cap U$.

Suppose $\frac{\partial f}{\partial x_1} \neq 0$ on $\Sigma \cap U$.

Set $y_1 = f(x)$; $y_2 = x_2, \dots, y_n = x_n$

$$\text{Jacobian} = \det \begin{bmatrix} \partial f/\partial x_1 & \partial f/\partial x_2 & \cdots & & \partial f/\partial x_n \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & & \cdots & 1 \end{bmatrix} = \frac{\partial f}{\partial x_1} \neq 0$$

<u>$k=2$</u>: $\Sigma \cap U = \{ f_1 = f_2 = 0 \}$; df_1 and df_2 lin ind on $\Sigma \cap U$

$\Sigma_1 = \{ f_1 = 0 \}$; $df_1 \neq 0$ so assume $\frac{\partial f}{\partial x_1} \neq 0$

$y_1 = \frac{\partial f}{\partial x_1}$, $y_j = x_j$, $j \geq 2$.

$\Sigma_1 = \{ y_1 = 0 \}$; $\Sigma_2 = \{ f_2 = 0 \} = \{ \tilde{f}_2 = 0 \}$

But df_2 and df_1 are l.i. so

$d\tilde{f}_2$ and $dy_1 = e_1$ are l.i.

assume $\partial \tilde{f}_2 / \partial y_2 \neq 0$, set

$\tilde{y}_1 = y_1$, $\tilde{y}_2 = \tilde{f}_2$, $\tilde{y}_j = y_j$ $j \geq 3$.

Check the Jacobian is not equal to zero

~~Hence~~. Hence we may always assume that locally

$$\Sigma = \{ x_1 = x_2 = \dots = x_k = 0 \}$$

Find vector fields tangent to Σ.

$$V = \sum_{j=1}^{n} a_j \partial_{x_j}$$

$$V x_1 = a_1 = 0 \quad \text{on } \Sigma.$$

$$a_1 = \sum_{l=1}^{k} a_{1l} x_l$$

$$V x_2 = a_2 = 0 \quad \text{on } \Sigma \qquad a_2 = \sum_{l=1}^{k} a_{2l} x_l.$$

Vector fields are spanned by. (with C^∞ coefficients)

$$x_1 \partial_{x_1}, x_2 \partial_{x_1}, \dots, x_k \partial_{x_1};$$

$$x_2 \partial_{x_1}, x_2 \partial_{x_2}, \dots, x_k \partial_{x_2}.$$

$$\vdots$$

$$x_1 \partial_{x_k}, x_2 \partial_{x_k}, \dots, x_k \partial_{x_k}.$$

$$\partial_{x_{k+1}}, \partial_{x_{k+2}}, \dots, \partial_{x_n}.$$

$\Sigma = \{ x_1 = x_2 = 0 \}$, v is tangent to Σ if and only if

$$v = a_1 x_1 \partial_{x_1} + a_2 x_2 \partial_{x_1} + b_1 x_1 \partial_{x_2} + b_2 x_2 \partial_{x_2} + \sum_{j=3}^{n} \alpha_j(x) \partial_{x_j}$$

<u>Exercise:</u> Let $U \cap \Sigma = \{ x_1 = \cdots = x_k = 0 \}$

Show that $u \in I^s(\Sigma) \iff \varphi \in C_0^\infty(U)$

$$x'^{\alpha'} D_{x'}^{\beta'} D_{x''}^{\gamma} (\varphi u) \in H^s(\mathbb{R}^n), \quad \alpha' \geq \beta'$$

$x' = (x_1, \ldots, x_k)$; $D_{x'} = (D_{x_1}, \ldots, D_{x_k})$.

$D_{x''} = (D_{x_{k+1}}, \ldots, D_{x_n})$.

$(x')^{\alpha'} = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_k^{\alpha_k}$

$(D_{x'})^{\beta'} = D_{x_1}^{\beta_1} \cdots D_{x_k}^{\beta_k}$

$\alpha_1 \geq \beta_1$, $\alpha_2 \geq \beta_2, \ldots, \alpha_k \geq \beta_k$.