INVERSE SCATTERING WITH DISJOINT SOURCE AND OBSERVATION SETS ON ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

RAPHAEL HORA AND ANTÔNIO SÁ BARRETO

Abstract. In this note, the scattering operator of an asymptotically hyperbolic manifold is only allowed to act on functions supported on the source set $\mathbb{R}\times\overline{\mathcal{O}}$, where $\mathcal{O}$ is an open subset of the boundary, and the resulting functions are then restricted to the observation set $\mathbb{R}\times\overline{\Gamma}$, where Γ is open. When Γ is the complement of the closure of $\mathcal{O}$, we call the corresponding operator the off-diagonal scattering operator with respect to $\mathcal{O}$. We prove that for a non-empty open proper subset $\mathcal{O}$, such that the intersection of its closure and its complement is not empty, the off-diagonal scattering operator with respect to $\mathcal{O}$ determines the manifold modulo isometries which are equal to the identity at the boundary. We also prove there is no analogue of the L^2 boundary controllability from an open subset of the boundary for radiation fields, which presents a possible obstacle to extending the inverse result for arbitrary disjoint subsets $\mathcal{O}$ and Γ.

1. Introduction

We continue our work on inverse scattering on asymptotically hyperbolic manifolds started in [23, 41]. In this paper, the scattering operator only acts on functions supported on the source set $\mathbb{R}\times\overline{\mathcal{O}}$, where $\mathcal{O}$ is an open subset of the boundary and the resulting functions are then restricted to the observation set $\mathbb{R}\times\overline{\Gamma}$, where Γ is open. We pose the problem of recovering the asymptotically hyperbolic manifold from this operator. The case where $\mathcal{O}$ and Γ are the whole boundary of the manifold was solved in [41], while the case where $\mathcal{O}$ is a proper subset, but $\mathcal{O}\cap\Gamma\neq\emptyset$ was solved in [23]. The next natural question is to consider the problem for arbitrary disjoint $\mathcal{O}$ and Γ. We show that this question has very serious additional difficulties. We also prove that if Γ is the complement of the closure of $\mathcal{O}$ and $\overline{\Gamma}\cap\overline{\mathcal{O}}\neq\emptyset$, the restriction of the scattering operator to $\mathcal{O}$ and Γ determine the manifold modulo isometries. See Theorem 1.1 below for the precise statement. The condition that $\overline{\mathcal{O}}$ intersects its complement says that $\mathcal{O}$ cannot be the union of whole connected components of the boundary of the manifold.

The scattering matrix is obtained by conjugation of the scattering operator with the Fourier transform, so this problem can be interpreted as an inverse problem for the scattering matrix at all energies acting between two disjoint open subsets. For fixed energy, the scattering matrix is a pseudo-differential operator, so the Schwartz kernel of the restriction of the scattering matrix to two disjoint open subsets is a smooth function. This particular result for the case where $\Gamma=\mathcal{O}^c$ case shows that the smooth part of the scattering matrix, when all energies are taken into account, determines the manifold modulo invariants. But the condition that the intersection of the closure of $\mathcal{O}$ and its complement is not empty is used to obtain information about the limit of a suitable function at one point of the diagonal and for one energy.

As far as we know, with the exception of situations where the inverse scattering problems can be reduced to a problem about the Dirichlet-to-Neumann map, such as the case of compactly

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supported perturbations of hyperbolic space, this is the first result about the determination of the manifold from off-diagonal information on the kernel of the scattering operator. We also show that there is no analogue of the L^2 boundary controllability from an open subset of the boundary for radiation fields. This presents a possible obstacle to extending Theorem 1.1 to arbitrary disjoint subsets $\mathcal{O}$ and Γ.

Throughout this paper $(\mathring{X}, g)$ denotes a C^∞ manifold equipped with a C^∞ Reimannian metric g. We shall assume that $X = \overline{\mathring{X}}$ is a C^∞ manifold with boundary and that ρ is a C^∞ function on X such that $\rho \geq 0$, $\rho^{-1}(0) = \partial X$ and $d\rho(0) \neq 0$. Such a function will be called a defining function of the boundary. We assume that $(X, \rho^2 g)$ is a C^∞ compact Riemannian manifold and furthermore, we will assume that if $h_0 = \rho^2 g|_{\partial X}$, then $|d\rho|_{\partial X}|_{h_0} = 1$. According to Mazzeo & Melrose [36], this guarantees that the sectional curvatures converge to -1 along any curve that goes towards the boundary. The manifold $(\mathring{X}, g)$ will be called an asymptotically hyperbolic manifold (AHM). The prototype of this class is the hyperbolic space $\mathring{X} = \{z \in \mathbb{R}^n : |z| < 1\}$ equipped with the metric $g_0 = 4(1 - |z|^2)^{-2} dz^2$ of constant curvature -1. Quotients of hyperbolic space by convex co-compact groups also fall in this category.

The metric g defines a conformal structure on ∂X and Graham [15] has shown that fixed a representative of $\rho^2 g|_{\partial X}$, there exists a unique function x defined in a collar neighborhood U_ε of ∂X such that

$$x^2 g = dx^2 + h(x) \text{ on } [0, \varepsilon) \times \partial X, \quad h(0) = x^2 g|_{\partial X}, \tag{1.1}$$

where $h(x)$ is a C^∞ family of metrics on ∂X for $x \in [0, \varepsilon)$. From now on we will use such the identification $U_\varepsilon \sim [0, \varepsilon)_x \times \partial X$. The function x is in principle defined only on a collar neighborhood of ∂X, but it can be extended to the whole manifold as a boundary defining function, but of course (1.1) only holds near ∂X.

The study of scattering theory of AHM started with the work of Fadeev, Fadeev & Pavlov [9, 14] and Lax & Phillips [32, 33], Agmon [1], Guillemin [19], and Perry [38, 39]. Mazzeo & Melrose [36] constructed a parametrix for the resolvent for the Laplacian on general AHM and used it to show that the resolvent continues meromorphically to the complex plane with the exception of a discrete set of points. Guillarmou [18] completed their program by showing that the resolvent can have essential singularities at the points excluded in the construction of Mazzeo and Melrose and by characterizing the metrics for which the resolvent continues to the entire complex plane. The work of Mazzeo and Melrose made it possible to study stationary scattering theory on AHM. Melrose [37] studied the scattering matrix on general AHM, while Guillopé [20] analyzed the scattering matrix on Riemann surfaces. Joshi and the second author [28] began the study of the inverse scattering theory on AHM. They showed that the scattering matrix at a fixed energy determines the Taylor series of the metric at infinity. Graham and Zworski [16] identified some conformal invariants in terms of residues of the scattering matrix. If an AHM manifold is also Einstein, Guillarmou and Sá Barreto [8] showed that the scattering matrix at one energy determines the manifold. Isozaki and Kurylev [25] and Isozaki, Kurylev and Lassas [24, 26, 27] studied inverse scattering on more general classes of manifolds, but they assume the metric is known (modulo isometries) in a neighborhood of one of the ends of the manifold.

Lax and Phillips [32, 33] studied the time dependent scattering theory in certain non-Euclidean spaces which included some examples of AHM. The second author [41] constructed the Friedlander radiation fields on general AHM and used it to prove that the scattering operator of an asymptotically hyperbolic manifold determines the manifold (including its topology and C^∞ structure) modulo isometries which are equal to the identity at the boundary. The current authors [23] showed that if the interiors of the source and observation sets intersect, then $(\mathring{X}, g)$ is determined modulo isometries which are equal to the identity at the intersection. Guillarmou and the second author [17] extended the result of [41] to the case of complex hyperbolic manifolds.

For non-compact manifolds, the scattering operator plays the role of the Dirichlet-to-Neumann Map (DTNM) for the wave equation. In fact one can show, see for example [37], that for compactly supported perturbations of the Euclidean, or hyperbolic spaces, the scattering operator determines the DTNM on a ball that contains the support of the perturbation. The surge of interest on inverse problems for the DTNM on compact manifolds with boundary started with the celebrated work of Calderón [7], see for example Uhlmann's survey [44] and references cited there for a historical account. The anisotropic Calderón's problem questions whether that a compact C^∞ Riemannian manifold X with boundary ∂X can determined modulo isometries from the DTNM for its Laplacian. This is a very difficult problem that has been studied by many people, and we again refer to Uhlmann's survey for a detailed account. We want to point out that Daudé, Kamran and Nicoleau [10, 11, 12, 13] have recently shown that the anisotropic Calderón problem in the case where the DTNM acts between two disjoint open subsets $\mathcal{O}$ and Γ of he boundary ∂X such that $\mathcal{O} \cap \Gamma = \emptyset$ and $\overline{\mathcal{O}} \cup \overline{\Gamma} \neq \partial X$ is false. They construct examples of conformal metrics and non-isometric metrics which have the same DTNM acting between $\mathcal{O}$ and Γ. It is interesting this work excludes precisely the case we discuss here. It is not clear the techniques of Daudé, Kamran and Nicoleau can be used to construct counter-examples in the time dependent problem discussed in this note.

The problem of determining a C^∞ Riemannian manifold modulo isometries from the DTNM for the wave equation was first studied by Belishev and Kurylev [4] using the boundary control method. This method relied on a unique continuation theorem proved by Tataru [43]. One can consult the book by Katchalov, Kurylev and Lassas [29] for a thorough account.

Lassas and Oksanen [30, 31] studied the question for the Dirichlet-to-Neumann map for the wave equation with source and data subsets having disjoint interiors. In [30] Lassas and Oksanen prove that if the closure of the source set and the closure of the observation set intersect, the manifold and the metric are determined modulo diffeomorphisms. In [31], they do not assume that closure of the sets intersect, but they have to assume that "enough information" from the source set arrives at the observation set, and this translates into the assumption that the wave equation is exactly controllable from the set of sources or the set of measurements or one of these sets satisfy the Hassell-Tao condition [21, 22]. Here we prove that L^2 boundary controllability for the wave equation from radiation fields restricted to any open subset of the boundary at infinity does not hold. This is somehow expected because of the results of Bardos, Lebeau and Rauch [2] – unlike the situation of compact manifolds with boundary, in the scattering case, bicharacteristics do not reflect at the boundary at infinity.

We from now on we assume $(\mathring{X}, g)$ is a C^∞ AHM. We use Δ_g to denote the Laplacian with respect to the metric g and we work with the wave operator associated with $\Delta_g - \frac{n^2}{4}$ –the factor $\frac{n^2}{4}$ serves to shift the continuous spectrum of Δ_g to $[0, \infty)$– and consider solutions of the Cauchy problem

$$\begin{gathered}(D_t^2 - \Delta_g - \frac{n^2}{4})u = 0, \text{ on } \mathbb{R}_\pm \times \mathring{X}, \\ u(0,z) = f_1, \quad D_t u(0,z) = f_2, \quad f_1, f_2 \in C_0^\infty(\mathring{X}).\end{gathered} \tag{1.2}$$

The spectrum of Δ_g is was studied by Mazzeo [35] and Mazzeo & Melrose [36], and by Bouclet [5]; it consists of two parts $\sigma_{pp} \cup \sigma_{ac}$. The point spectrum σ_{pp} is finite and contained in $(0, \frac{n^2}{4})$, while the continuous spectrum $\sigma_{ac} = [\frac{n^2}{4}, \infty)$. The conserved energy

(1.3)
$$E(u, \partial_t u)(t) = \int_X \left(|du(t)|^2 - \frac{n^2}{4}|u(t)|^2 + |\partial_t u(t)|^2 \right) \mathrm{dvol}_g = \int_X \left(|df_1|^2 - \frac{n^2}{4}|f_1|^2 + |f_2|^2 \right) \mathrm{dvol}_g$$

is only coercive only when projected onto $L^2_{ac}(X)$. We then define the projector

$$\begin{gathered} \mathcal{P}_{ac} : L^2(X) \longrightarrow L^2_{ac}(X) \\ f \longmapsto f - \sum_{j=1}^{N} \langle f, \phi_j \rangle \phi_j, \end{gathered}$$

where $\{\phi_j, 1 \leq j \leq N\}$ are the eigenfunctions of Δ_g, We define the energy space $E_{ac}(X)$ to be the closure of $(\varphi, \psi) \in C_0^\infty(\mathring{X}) \times C_0^\infty(\mathring{X})$ that are orthogonal to the eigenfunctions of Δ_g and for which with norm

$$||(\phi, \psi)||_E = \int_X (|\psi|^2 + |d_g\varphi|^2 - \frac{n^2}{4}|\varphi|^2) \, d\,\mathrm{vol}_g < \infty.$$

This definition makes sense in view of Corollary 6.3 of [41]. We will also need to introduce the space

$$\begin{gathered} \dot{H}^1_{\mathrm{ac}}(X) = \{\varphi : 0 \leq \int_X (|d_g\varphi|^2 - \frac{n^2}{4}|\varphi|^2) \, d\,\mathrm{vol}_g < \infty\}, \\ \text{with norm } ||\varphi||^2_{\dot{H}^1_{\mathrm{ac}}} = \int_X (|d_g\varphi|^2 - \frac{n^2}{4}|\varphi|^2) \, d\,\mathrm{vol}_g \,. \end{gathered}$$

We shall fix a product decomposition $z = (x, y) \in U_\varepsilon \sim [0, \varepsilon) \times \partial X$, in which (1.1) holds and our definitions will depend on such choice. One can remove this dependence by working on appropriate vector bundles, but we will not do this here.

Let u satisfy (1.2) with initial data $(f_1, f_2) \in (C_0^\infty(\mathring{X}) \times C_0^\infty(\mathring{X})) \cap E_{\mathrm{ac}}(X)$. It was shown in [41] that

(1.4)
$$V_\pm(x, s_\pm, y) = x^{-n/2} u(s_\pm \pm \log x, x, y) \in C^\infty([0, \varepsilon)_x \times \mathbb{R}_{s_\pm} \times \partial X).$$

The forward and backward radiation fields of $(f_1, f_2) \in C_0^\infty(\mathring{X}) \times C_0^\infty(\mathring{X})$ are defined to be

(1.5)
$$\begin{gathered} \mathcal{R}_\pm : C_0^\infty(\mathring{X}) \times C_0^\infty(\mathring{X}) \longrightarrow C^\infty(\mathbb{R} \times \partial X), \\ \mathcal{R}_\pm(f_1, f_2)(s, y) = D_{s_\pm} V_\pm(0, s_\pm, y) \end{gathered}$$

and it was shown in [41] that $\mathcal{R}_\pm$ extend to unitary operators

(1.6)
$$\begin{gathered} \mathcal{R}_\pm : E_{ac}(X) \longrightarrow L^2(\mathbb{R} \times \partial X), \\ (f_1, f_2) \longmapsto \mathcal{R}_\pm(f_1, f_2), \end{gathered}$$

where the measure on ∂X is the one induced by the metric h_0 defined in (1.1).

The scattering operator is defined to be the map

(1.7)
$$\begin{gathered} \mathcal{S} : L^2(\mathbb{R} \times \partial X) \longrightarrow L^2(\mathbb{R} \times \partial X), \\ \mathcal{S} = \mathcal{R}_+ \circ \mathcal{R}_-^{-1}. \end{gathered}$$

Since $\mathcal{R}_\pm$ are unitary, $\mathcal{S}$ is unitary in $L^2(\partial X\times\mathbb{R})$ and commutes with translations in the s variable. The second author and Wang [42] showed that $\mathcal{S}$ is a Fourier integral operator that quantizes the scattering relation.

If $U\subset\partial X$ is an open subset, we shall denote

$$L^2(\mathbb{R}\times U)=\{F\in L^2(\mathbb{R}\times\partial X):\ F \text{ is supported in } \mathbb{R}\times\overline{U}\}.$$

If $\mathcal{O}_j\subset\partial X$, $j=1,2$ are open subsets of ∂X. We define the scattering operator with sources placed on $\mathcal{O}_1$ and measurements made on $\mathcal{O}_2$ as the map

$$\begin{aligned}\mathcal{S}_{\mathcal{O}_2\mathcal{O}_1}: L^2(\mathbb{R}\times\mathcal{O}_1)&\longrightarrow L^2(\mathbb{R}\times\mathcal{O}_2)\\ F&\longmapsto(\mathcal{S}F)|_{\mathbb{R}\times\overline{\mathcal{O}}_2}\end{aligned}\tag{1.8}$$

Given a non-empty open subset $\mathcal{O}\Subset\partial X$, we defined the *off-diagonal scattering operator* with respect to $\mathcal{O}$ to be the operator $\mathcal{S}_{\mathcal{O}^c\mathcal{O}}$.

The following is the main result of this paper:

Theorem 1.1. *Let $(\mathring{X}_j,g_j)$, $j=1,2$, be connected AHM. Suppose that $M=\partial X_1=\partial X_2$ and that the conformal representatives $h_{0,j}=\rho_j^2g_j|_M$ satisfy $h_0=h_{0,1}=h_{0,2}$. Let $\mathcal{S}_j$ denote the scattering operator with respect to $(\mathring{X}_j,g_j)$, for the particular choice of h_0. Let $\mathcal{O}\subset M$ be a non-empty open subset such that $\mathcal{O}^c=M\setminus\overline{\mathcal{O}}$ is not empty. Moreover assume that $\overline{\mathcal{O}}\cap\overline{\mathcal{O}^c}\neq\emptyset$. If the corresponding off-diagonal scattering operators satisfy $\mathcal{S}_{1,\mathcal{O}^c\mathcal{O}}=\mathcal{S}_{2,\mathcal{O}^c\mathcal{O}}$, then there exists a diffeomorphism $\Psi:(X_1,g_1)\longrightarrow(X_2,g_2)$ such that $\Psi|_M=\mathrm{id}$, and $\Psi^*g_2=g_1$.*

Notice that Theorem 1.1 can be interpreted in the following way:

Theorem 1.2. *Let $(\mathring{X}_j,g_j,)$ $j=1,2$, be connected AHM. Suppose that $M=\partial X_1=\partial X_2$ and that the conformal representatives $h_{0,j}=\rho_j^2g_j|_M$ satisfy $h_0=h_{0,1}=h_{0,2}$. Let $\mathcal{O}\subset M$ be a non-empty open subset such that $\mathcal{O}^c=M\setminus\overline{\mathcal{O}}$ is not empty. Moreover assume that $\overline{\mathcal{O}}\cap\overline{\mathcal{O}^c}\neq\emptyset$. Let $\mathcal{S}_j$ denote the scattering operator with respect to $(\mathring{X}_j,g_j)$, for this particular choice of h_0 and let $K_{\mathcal{S}_j}$ denote the Schwartz kernel of $\mathcal{S}_j$, $j=1,2$. Suppose that $K_{\mathcal{S}_1}(s,s',y,y')=K_{\mathcal{S}_2}(s,s',y,y')$ as distributions on $\mathbb{R}\times\mathbb{R}\times\mathcal{O}^c\times\mathcal{O}$, then there exists a diffeomorphism $\Psi:(X_1,g_1)\longrightarrow(X_2,g_2)$ such that $\Psi|_M=\mathrm{id}$ and $\Psi^*g_2=g_1$.*

The analogue of this result for $\mathcal{S}_{\partial X\partial X}=\mathcal{S}$ was proved in [41] and the case $\mathcal{S}_{\mathcal{O}\mathcal{O}}$ was proved in [23]. In Proposition 2.10, we will show that $\mathcal{S}_{\mathcal{O}^c\mathcal{O}}$ determines $\mathcal{S}_{\partial X\partial X}$. Therefore, the proof of Theorem 1.1 is reduced to an application of the result of [41]. We remark that the results of [23, 41] were based on an adaptation of the boundary control method of Belishev [3], Belishev & Kurylev [4] and Tataru [43].

2. The Local Support Theorem and its Consequences

The key ingredient in the proof of Theorem 1.1 is the following support theorem proved in [23]:

Theorem 2.1. *Let $\mathcal{O}\subset\partial X$ be a nonempty open subset, let $f\in L^2_{ac}(X)$ and let $s_0\in\mathbb{R}$. Let $\varepsilon>0$ be such that (1.1) holds in $(0,\varepsilon)\times\partial X$, and let $\overline{\varepsilon}=\min\{\varepsilon,e^{s_0}\}$. Then $\mathcal{R}_+(0,f)(s,y)=0$ in $\{s<s_0,\ y\in\mathcal{O}\}$ if and only if for every $z=(x,y)\in(0,\overline{\varepsilon})\times\mathcal{O}=W_\varepsilon$*

$$d_g(z,supp\ f)>\log(\frac{e^{s_0}}{x}),\tag{2.1}$$

where d_g denotes the distance function with respect to the metric g, and $supp\ f$ denotes the support of f.

The following result is a consequence of Theorem 2.1 which is fundamental in the proof of Theorem 1.1:

Corollary 2.2. *Let (X, g) be a connected AHM and let $\mathcal{O} \subset \partial X$ be open, $\mathcal{O} \neq \emptyset$. If $f \in L^2_{ac}(X)$ and $\mathcal{R}_+(0, f)(s, y) = 0$ in $\mathbb{R} \times \mathcal{O}$, then $f = 0$. Similarly, if $(h, 0) \in E_{ac}(X)$ and $\mathcal{R}_+(h, 0)(s, y) = 0$ in $\mathbb{R} \times \mathcal{O}$, then $h = 0$.*

Proof. If $\mathcal{R}_+(0, f)(s, y) = 0$ in $\mathbb{R} \times \mathcal{O}$, then for every $z = (x, y) \in W_{\bar{\varepsilon}}$, $d(z, \operatorname{supp} f) > s_0 - \log x$ for every s_0. Since the distance between any two points in the interior of X is finite, it follows that $\operatorname{supp} f$ is empty and $f = 0$.

Suppose $F = \mathcal{R}_+(h, 0)(s, y) = 0$ in $\mathbb{R} \times \mathcal{O}$. By taking convolution of F with $\phi \in C_0^\infty(\mathbb{R})$, even, we may assume that $(\Delta_g - \frac{n^2}{4})^k h \in L^2_{ac}(X)$ for every $k \geq 0$, see section 3.1 of [23] for a proof. Then $\partial_s F = \mathcal{R}_+ \left(0, (\Delta_g - \frac{n^2}{4})h\right)(s, y) = 0$ in $\mathbb{R} \times \mathcal{O}$. But as we have shown, this implies that $(\Delta_g - \frac{n^2}{4})h = 0$, and hence $\partial_s F = 0$. Since $F \in L^2(\mathbb{R} \times \partial X)$, it follows that $F = 0$ and hence $h = 0$. □

This result allows us to define the following operators: Let $\psi \in L^2_{\rm ac}(X)$ and $\varphi \in \dot{H}^1_{\rm ac}(X)$ be such that $\mathcal{R}_+(\varphi, \psi)$ is supported in $\mathbb{R} \times \overline{\mathcal{O}}$, and let

$$
\begin{aligned}
T_{\mathcal{O}} : L^2_{ac}(X) &\longrightarrow \dot{H}^1_{\rm ac}(X) & \qquad T_{\mathcal{O}}^{-1} : \dot{H}^1_{\rm ac}(X) &\longrightarrow L^2_{\rm ac}(X) \\
\psi &\longmapsto \varphi & \varphi &\longmapsto \psi,
\end{aligned} \tag{2.2}
$$

First we show these are densely defined and closed operators

Lemma 2.3. *Let $(\mathring{X}, g)$ be a connected AHM manifold. Let $\mathcal{O} \subset \partial X$ be a nonempty open subset such that $\mathcal{O}^c = \partial X \setminus \overline{\mathcal{O}} \neq \emptyset$. For any $\psi \in L^2_{ac}(X)$ there exists at most one $\varphi \in \dot{H}^1_{\rm ac}(X)$ such that $\mathcal{R}_+(\varphi, \psi)$ is supported in $\mathbb{R} \times \overline{\mathcal{O}}$, and for any $\varphi \in \dot{H}^1_{\rm ac}(X)$ there exists at most one $\psi \in L^2(X)$ such that $\mathcal{R}_+(\varphi, \psi)$ is supported in $\mathbb{R} \times \overline{\mathcal{O}}$. Let*

(2.3)

$$
\begin{aligned}
\mathcal{C}(\mathcal{O}) &= \{\psi \in L^2_{ac}(X) : \text{ there exists } \varphi \in \dot{H}^1_{\rm ac}(X) \text{ such that } \mathcal{R}_+(\varphi, \psi)(s, y) \text{ is supported in } \mathbb{R} \times \overline{\mathcal{O}}\}, \\
\mathcal{E}(\mathcal{O}) &= \{\varphi \in \dot{H}^1_{\rm ac}(X) : \text{ there exists } \psi \in L^2_{ac}(X) \text{ such that } \mathcal{R}_+(\varphi, \psi)(s, y) \text{ is supported in } \mathbb{R} \times \overline{\mathcal{O}}\}.
\end{aligned}
$$

Then $\mathcal{C}(\mathcal{O})$ is the domain of $T_{\mathcal{O}}$ and is dense in $L^2_{ac}(X)$. Similarly, $\mathcal{E}(\mathcal{O})$ is the domain of $T_{\mathcal{O}}^{-1}$ and is dense in $\dot{H}^1_{\rm ac}(X)$.

Proof. First, if $\mathcal{R}_+(\varphi_1, \psi)$ and $\mathcal{R}_+(\varphi_2, \psi)$ are supported in $\mathbb{R} \times \overline{\mathcal{O}}$, then $\mathcal{R}_+(\varphi_1 - \varphi_2, 0)$ is supported in $\mathbb{R} \times \overline{\mathcal{O}}$, but this implies that $\mathcal{R}_+(\varphi_1 - \varphi_2, 0) = 0$ in $\mathbb{R} \times \mathcal{O}^c$, and so Corollary 2.2 implies that $\varphi_1 = \varphi_2$. The uniqueness of ψ for a fixed φ follows from the same argument.

We will prove the density of $\mathcal{C}(\mathcal{O})$ in $L^2_{ac}(X)$, the proof of the density of $\mathcal{E}(\mathcal{O})$ in $\dot{H}^1_{\rm ac}(X)$ is identical. Let $v \in L^2_{ac}(X)$ be such that $\langle v, \psi \rangle_{L^2(X)} = 0$ for all $\psi \in \mathcal{C}(\mathcal{O})$. Then, since $\mathcal{R}_+$ is unitary this implies that in particular

$$
0 = \langle v, \psi \rangle_{L^2(X)} = \langle \mathcal{R}_+(0, v), \mathcal{R}_+(\varphi, \psi) \rangle_{L^2(\mathbb{R} \times \partial X)}, \text{ for all } \mathcal{R}_+(\varphi, \psi) \text{ supported in } \mathbb{R} \times \overline{\mathcal{O}}.
$$

It follows that $\mathcal{R}_+(0, v) = 0$ on $\mathbb{R} \times \mathcal{O}$ and by Corollary 2.2, $v = 0$. □

Next we show that if $\mathcal{O}$ is a proper subset of ∂X, $T_{\mathcal{O}}$ and $T_{\mathcal{O}}^{-1}$ are unbounded.

Lemma 2.4. *Let $(\mathring{X}, g)$ be an AHM. If $\mathcal{O} \subset \partial X$ and $\Gamma \subset \partial X$ are nonempty open subsets such that $\overline{\mathcal{O}} \cap \overline{\Gamma} = \emptyset$, and $\partial X \setminus (\overline{\mathcal{O}} \cup \overline{\Gamma})$ is not empty, then $\mathcal{C}(\mathcal{O}) \cap \mathcal{C}(\Gamma) = \emptyset$ and $\mathcal{E}(\mathcal{O}) \cap \mathcal{E}(\Gamma) = \emptyset$, where $\mathcal{C}(\bullet)$ and $\mathcal{E}(\bullet)$, $\bullet = \mathcal{O}, \Gamma$ are the spaces defined in (2.3).*

Proof. Suppose $\psi \in \mathcal{C}(\mathcal{O}) \cap \mathcal{C}(\Gamma)$, $\psi \neq 0$, then there exist φ and $\tilde{\varphi}$ such that

$$\mathcal{R}_+(\varphi, \psi) \text{ is supported in } \overline{\mathcal{O}}, \text{ and } \mathcal{R}_+(\tilde{\varphi}, \psi) \text{ is supported in } \overline{\Gamma},$$

but then $\mathcal{R}_+(\varphi - \tilde{\varphi}, 0)$ is supported in $\overline{\mathcal{O}} \cup \overline{\Gamma}$. Since $\partial X \setminus (\overline{\mathcal{O}} \cup \overline{\Gamma})$ is a non-empty open subset, Corollary 2.2 implies that $\varphi = \tilde{\varphi}$, but this is not possible since $\overline{\mathcal{O}} \cap \overline{\Gamma} = \emptyset$. $\square$

Proposition 2.5. *Let $(\mathring{X}, g)$ be an AHM. If $\mathcal{O} \subset \partial X$ is a non-empty open subset such that $\mathcal{O}^c = \partial X \setminus \overline{\mathcal{O}} \neq \emptyset$, then the operators $T_{\mathcal{O}}$ and $T_{\mathcal{O}}^{-1}$ are closed and unbounded.*

Proof. Let $\Gamma \Subset \mathcal{O}^c$ be an open subset. Therefore $\partial X \setminus (\overline{\mathcal{O}} \cup \overline{\Gamma}) \neq \emptyset$. If $T_{\mathcal{O}}$ were defined everywhere, then its domain $\mathcal{C}(\mathcal{O}) = L^2_{\text{ac}}(X)$ and in particular $\mathcal{C}(\mathcal{O}) \cap \mathcal{C}(\Gamma) = \mathcal{C}(\Gamma)$, which would contradict Lemma 2.4.

If $\psi_n \in \mathcal{C}(\mathcal{O})$, $\psi_n \to \psi$ in $L^2_{\text{ac}}(X)$ and $\varphi_n = T_{\mathcal{O}}\psi_n \to \varphi$ in $\dot{H}^1_{\text{ac}}(X)$, then $\mathcal{R}_+(\varphi_n, \psi_n) \to \mathcal{R}_+(\varphi, \psi)$ in $L^2(\mathbb{R} \times \partial X)$. Since $\mathcal{R}_+(\varphi_n, \psi_n)$ are supported on $\mathbb{R} \times \overline{\mathcal{O}}$, then $\mathcal{R}_+(\varphi, \psi)$ is supported on $\mathbb{R} \times \overline{\mathcal{O}}$ and therefore $\varphi = T_{\mathcal{O}}\psi$.

This argument also applies in the case of $T_{\mathcal{O}}^{-1}$. $\square$

There is an important consequence of Proposition 2.5, which implies that even though $\mathcal{R}_\pm$ are unitary, one cannot control the norm of $\psi \in L^2_{\text{ac}}(X)$ from the L^2 norm of $\mathcal{R}_\pm(0, \psi)|_{\mathbb{R} \times \mathcal{O}}$ for any open subset $\mathcal{O} \subset \partial X$ such that $\partial X \setminus \overline{\mathcal{O}} \neq \emptyset$.

Proposition 2.6. *Let $\mathcal{O} \subset \partial X$ be an open subset such that $\mathcal{O}^c = \partial X \setminus \overline{\mathcal{O}}$ is not empty. Then there exist sequences $\psi_n \in L^2_{ac}(X)$ and $\varphi_n \in \dot{H}^1_{\text{ac}}(X)$ such that*

$$\begin{aligned} &||\psi_n||_{L^2(X)} = 1, \quad ||\mathcal{R}_+(0, \psi_n)|_{\mathbb{R} \times \overline{\mathcal{O}}}||_{L^2} \leq \frac{1}{n}, \\ &||\varphi_n||_{\dot{H}^1_{\text{ac}}(X)} = 1, \quad ||\mathcal{R}_+(\varphi_n, 0)|_{\mathbb{R} \times \overline{\mathcal{O}}}||_{L^2} \leq \frac{1}{n}. \end{aligned} \tag{2.4}$$

Proof. Let $\Gamma \Subset \partial X \setminus \overline{\mathcal{O}}$, then $\partial X \setminus (\overline{\mathcal{O}} \cup \overline{\Gamma})$ is a non-empty open subset of ∂X. Let T_Γ be the linear operator defined by (2.2). As observed above, T_Γ is an unbounded operator and so there exists a sequence $\psi_n \in L^2_{\text{ac}}(X)$ such that

$$||\psi_n||_{L^2_{\text{ac}}(X)} = 1, \quad ||T_\Gamma \psi_n||_{\dot{H}^1_{\text{ac}}(X)} \geq n.$$

But, on the other hand, from the definition of T_Γ, $\mathcal{R}_+(T_\Gamma\psi_n, \psi_n) = 0$ in $\mathbb{R} \times \mathcal{O}$. Therefore

$$||\mathcal{R}_+(T_\Gamma\psi_n, 0)|_{\mathbb{R} \times \overline{\mathcal{O}}}||_{L^2(\mathbb{R} \times \partial X)} = || - \mathcal{R}_+(0, \psi_n)|_{\mathbb{R} \times \overline{\mathcal{O}}}||_{L^2(\mathbb{R} \times \partial X)} \leq ||\psi_n||_{L^2_{\text{ac}}(X)} = 1.$$

If one divides this equation by $||T_\Gamma\psi_n||_{\dot{H}^1_{\text{ac}}(X)} \geq n$, it follows that

$$||\mathcal{R}_+(\frac{T_\Gamma\psi_n}{||T_\Gamma\psi_n||_{L^2(X)}}, 0)|_{\mathbb{R} \times \overline{\mathcal{O}}}||_{L^2(\mathbb{R} \times \partial X)} \leq \frac{||\psi_n||_{L^2_{\text{ac}}(X)}}{||T_\Gamma\psi_n||_{\dot{H}^1_{\text{ac}}(X)}} \leq \frac{1}{n}.$$

Thus $\varphi_n = \frac{T_\Gamma\psi_n}{||T_\Gamma\psi_n||_{L^2(X)}}$ satisfies the second part of (2.4). One can use T_Γ^{-1} to construct the sequence $\psi_n \in L^2_{\text{ac}}(X)$ that satisfies (2.4). $\square$

One can interpret this as the lack of L^2 boundary controllability from an open subset of the boundary for radiation fields. In other words, for any open set $\mathcal{O} \subset \partial X$ such that $\overline{\mathcal{O}} \neq X$, one cannot have

$$||\mathcal{R}_+(0,\psi)|_{\mathbb{R}\times\overline{\mathcal{O}}}||_{L^2(\mathbb{R}\times\partial X)} \geq C||\psi||_{L^2(X)} \text{ for all } \psi \in L^2_{ac}(X),$$
$$||\mathcal{R}_+(\varphi,0)|_{\mathbb{R}\times\overline{\mathcal{O}}}||_{L^2(\mathbb{R}\times\partial X)} \geq C||\varphi||_{\dot{H}^1_{\mathrm{ac}}(X)} \text{ for all } \varphi \in \dot{H}^1_{\mathrm{ac}}(X).$$

The necessary and sufficient geometric condition for boundary controllability from an open subset of the boundary of a compact Riemannian manifold was established by Bardos, Lebeau and Rauch [2] and by Burq and Gérard [6] and it says that every generalized bicharacteristic of the wave operator intersects this set at a non-diffractive point. In our setting geodesics do not reflect and it somehow explains why one should not expect to be able to control the wave equation from the radiation fields restricted to an open subset of ∂X.

One can also interpret this result as

Corollary 2.7. *Let $(\mathring{X}, g)$ be an AHM. Let $\mathcal{O}, \Gamma \Subset \partial X$ be disjoint open subsets. The operator $\mathcal{S}_{\mathcal{O}\Gamma}$ defined in* (1.8) *is injective, has dense range, but it is not onto.*

Proof. Suppose $F \in L^2(\mathbb{R} \times \partial X)$ is supported in $\mathbb{R} \times \overline{\Gamma}$. Let $(\varphi, \psi) \in E_{\mathrm{ac}}(X)$ be such that $F = \mathcal{R}_-(\varphi,\psi)$. Since F is supported in $\mathbb{R}\times\Gamma$, so is $F^*(s,y) = F(-s,y)$. But we know that $F^*(s,y) = \mathcal{R}_+(-\varphi,\psi)$ and therefore, since Γ and $\mathcal{O}$ are disjoint, $\mathcal{R}_+(-\varphi,\psi)|_{\mathbb{R}\times\mathcal{O}} = 0$. On the other hand $\mathcal{S}_{\mathcal{O}\Gamma}F = \mathcal{R}_+(\varphi,\psi)|_{\mathbb{R}\times\mathcal{O}}$, and hence $\mathcal{S}_{\mathcal{O}\Gamma}F = 2\mathcal{R}_+(0,\psi)|_{\mathbb{R}\times\mathcal{O}}$. If $\mathcal{S}_{\mathcal{O}\Gamma}F = 0$, it would follow that $\mathcal{R}_+(0,\psi)|_{\mathbb{R}\times\mathcal{O}} = 0$, and in view of Corollary 2.2, $\psi = 0$. But then $F = \mathcal{R}_-(\varphi, 0)$ is supported in $\mathbb{R}\times\overline{\Gamma}$, and hence again by Corollary 2.2, $F = 0$. So $\mathcal{S}_{\mathcal{O}\Gamma}$ is injective.

Suppose there exists $G = \mathcal{R}_+(f,h) \in L^2(\mathbb{R}\times\mathcal{O})$ such that

$$\langle G, \mathcal{S}_{\mathcal{O}\Gamma}F\rangle_{L^2(\mathbb{R}\times\mathcal{O})} = 0 \text{ for all } F \in L^2(\mathbb{R}\times\Gamma).$$

If $F = \mathcal{R}_+(\varphi,\psi)$, then $\frac{1}{2}\mathcal{S}_{\mathcal{O}\Gamma}F^* = \mathcal{R}_+(0,\psi)|_{\mathbb{R}\times\mathcal{O}}$. Therefore,

$$\langle \mathcal{R}_+(f,h), \mathcal{R}_+(0,\psi)\rangle_{L^2(\mathbb{R}\times\mathcal{O})} = \langle \psi, h\rangle_{L^2(X)} = 0 \text{ for all } \psi \in \mathcal{C}(\mathcal{O}).$$

Since $\mathcal{C}(\mathcal{O})$ is dense, this implies that $h = 0$. But then $G = \mathcal{R}_+(f,0)$ is supported in $\mathbb{R}\times\overline{\mathcal{O}}$. But we know from Corollary 2.2 this implies that $G = 0$. Therefore the range of $\mathcal{S}_{\mathcal{O}\Gamma}$ is dense.

If $\mathcal{S}_{\mathcal{O}\Gamma}$ were onto, then it would be invertible, by the open mapping theorem. This would imply that for every $F \in L^2(\mathbb{R}\times\Gamma)$,

$$||\mathcal{S}_{\mathcal{O}\Gamma}F||_{L^2(\mathbb{R}\times\partial X)} = ||2\mathcal{R}_+(0,\psi)|_{\mathbb{R}\times\overline{\mathcal{O}}}||_{L^2(\mathbb{R}\times\overline{\mathcal{O}})} \geq C||F||_{L^2(\mathbb{R}\times\partial X)} \geq C||\psi||.$$

So in particular,

$$||2\mathcal{R}_+(0,\psi)|_{\mathbb{R}\times\partial X}||_{L^2(\mathbb{R}\times\overline{\mathcal{O}})} \geq C||\psi||, \text{ for all } \psi \in \mathcal{C}(\Gamma).$$

Since $\mathcal{C}(\Gamma)$ is dense in $L^2(X)$. This would hold for every $\psi \in L^2_{ac}(X)$, which would contradict Proposition 2.6. $\square$

Next we analyze the adjoints of $T_{\mathcal{O}}$ and $T_{\mathcal{O}}^{-1}$.

Proposition 2.8. *Let $(\mathring{X}, g)$ be an AHM and let $\mathcal{O} \subset \partial X$ be such that $\mathcal{O}^c = \partial X \setminus \overline{\mathcal{O}}$ is not empty. Then $T_{\mathcal{O}}^* = -T_{\mathcal{O}^c}^{-1}$ and $T_{\mathcal{O}}^{-1*} = -T_{\mathcal{O}^c}$.*

Proof. Let $f \in \dot{H}^1_{\mathrm{ac}}(X)$ be such that there exists $f^* \in L^2_{ac}(X)$ such that

$$\langle T_{\mathcal{O}}\psi, f\rangle_{\dot{H}^1_{\mathrm{ac}}(X)} = \langle \psi, f^*\rangle_{L^2(X)}, \text{ for all } \psi \in \mathcal{C}(\mathcal{O}).$$

But since $\mathcal{R}_+$ is an isometry, and by definition of $T_{\mathcal{O}}$, we arrive at two identities:

$$\langle \psi, f^* \rangle_{L^2_{\mathrm{ac}}(X)} = \langle \mathcal{R}_+(T_{\mathcal{O}}\psi, \psi), \mathcal{R}_+(0, f^*) \rangle_{L^2(\mathbb{R}\times\partial X)},$$
$$\langle T_{\mathcal{O}}\psi, f \rangle_{\dot{H}^1_{\mathrm{ac}}(X)} = \langle \mathcal{R}_+(T_{\mathcal{O}}\psi, \psi), \mathcal{R}_+(f, 0) \rangle_{L^2(\mathbb{R}\times\partial X)}.$$

So we conclude that

$$\langle \mathcal{R}_+(T_{\mathcal{O}}\psi, \psi), \mathcal{R}_+(f, -f^*) \rangle = 0 \text{ for all } \psi \in \mathcal{C}(\mathcal{O}).$$

This of course implies that $\mathcal{R}_+(f, -f^*) = 0$ in $\mathbb{R} \times \mathcal{O}$, and therefore, $\mathcal{R}_+(f, -f^*)$ is supported in $\mathbb{R} \times \overline{\mathcal{O}^c}$. By definition, it follows that $f \in \mathcal{E}(\mathcal{O}^c)$, $f^* \in \mathcal{C}(\mathcal{O}^c)$ and $f^* = -T_{\mathcal{O}^c}^{-1} f$.

□

The Proposition 2.10 below reduces the proof of Theorem 1.1 to an application of the result of the second author [41] which states that $(\mathring{X}, g)$ is determined by $\mathcal{S}$ modulo isometries of (X, g) that fix ∂X. In the proof of Proposition 2.10 we need the following well known result due to Von Neumann, see for example section 118 of [40]:

Lemma 2.9. *Let H_1 and H_2 be Banach spaces and let $T : H_1 \longrightarrow H_2$ be a densely defined closed operator with domain $\mathcal{D}(T)$. Let T^* denote the adjoint of T and let $\mathcal{D}(T^*)$ denote the domain of T^*. Then the subspace $\mathcal{U} = \{h \in \mathcal{D}(T) : Th \in \mathcal{D}(T^*)\}$ is dense in H_1 and $T^*T : H_1 \longrightarrow H_1$ is self-adjoint. Moreover $B = (I + T^*T)^{-1} : H_1 \longrightarrow H_1$ is a bounded operator and $||B|| \leq 1$.*

If we apply this Lemma to $T_{\mathcal{O}} : L^2_{\mathrm{ac}}(X) \longrightarrow \dot{H}^1_{\mathrm{ac}}(X)$, where $\mathcal{O} \subset \partial X$ is open and $\partial X \setminus \overline{\mathcal{O}} \neq \emptyset$ and if we define $v = (I + T^*_{\mathcal{O}} T_{\mathcal{O}})^{-1}\psi$, then, $\psi = (I + T^*_{\mathcal{O}} T_{\mathcal{O}})v$ and one can use this to write

$$\mathcal{R}_+(0, \psi) = \mathcal{R}_+(T_{\mathcal{O}} v, v) - \mathcal{R}_+(T_{\mathcal{O}} v, -T^*_{\mathcal{O}} T_{\mathcal{O}} v). \tag{2.5}$$

By definition $\mathcal{R}_+(T_{\mathcal{O}} v, v)$ is supported in $\mathbb{R} \times \overline{\mathcal{O}}$ and since $T^*_{\mathcal{O}} = -T^{-1}_{\mathcal{O}^c}$, $\mathcal{R}_+(-T_{\mathcal{O}} v, T^*_{\mathcal{O}} T_{\mathcal{O}} v)$ is supported in $\mathbb{R} \times \overline{\mathcal{O}}^c$. This shows that $\mathcal{E}(\mathcal{O}) \cap \mathcal{E}(\mathcal{O}^c)$ contains the set $\{T_{\mathcal{O}} v,\ v \in L^2_{\mathrm{ac}}(X)\}$, which according to Lemma 2.9 is dense in $\dot{H}^1_{\mathrm{ac}}(X)$. This is in contrast with Lemma 2.4 which shows that $\mathcal{E}(\mathcal{O}) \cap \mathcal{E}(\Gamma) = \emptyset$, provided $\overline{\mathcal{O}} \cup \overline{\Gamma} \neq \partial X$.

Proposition 2.10. *Let $(\mathring{X}, g)$ be an AHM. Let $\mathcal{O} \subset \partial X$ be an open subset such that $\mathcal{O}^c = \partial X \setminus \overline{\mathcal{O}}$ is not empty and assume that $\overline{\mathcal{O}} \cap \overline{\mathcal{O}^c} \neq \emptyset$. Let $\mathcal{S}_{\mathcal{O}^c\mathcal{O}}$ be the scattering operator with sources on $\mathcal{O}$ and data on $\mathcal{O}^c$ defined in* (1.8). *Then $\mathcal{S}_{\mathcal{O}^c\mathcal{O}}$ determines $\mathcal{S}_{\mathcal{O}\mathcal{O}^c}$, $\mathcal{S}_{\mathcal{O}^c\mathcal{O}^c}$ and $\mathcal{S}_{\mathcal{O}\mathcal{O}}$. In particular it determines $\mathcal{S} = \mathcal{S}_{\mathcal{O}\mathcal{O}} + \mathcal{S}_{\mathcal{O}^c\mathcal{O}} + \mathcal{S}_{\mathcal{O}\mathcal{O}^c} + \mathcal{S}_{\mathcal{O}^c\mathcal{O}^c}$.*

Proof. First we check that $\mathcal{S}_{\mathcal{O}^c\mathcal{O}}$ determines $\mathcal{S}_{\mathcal{O}\mathcal{O}^c}$. Let $F = \mathcal{R}_+(f, h)$ be supported in $\mathbb{R} \times \overline{\mathcal{O}}$ and let $G = \mathcal{R}_+(\varphi, \psi)$ be supported in $\mathbb{R} \times \overline{\mathcal{O}}^c$. Then $F^* = \mathcal{R}_-(-f, h)$ and $G^* = \mathcal{R}_-(-\varphi, \psi)$ and so

$$\mathcal{S}_{\mathcal{O}^c\mathcal{O}} F^* = 2\mathcal{R}_+(0, h)|_{\mathbb{R}\times\overline{\mathcal{O}}^c} \text{ and } \mathcal{S}_{\mathcal{O}\mathcal{O}^c} G^* = 2\mathcal{R}_+(0, \psi)|_{\mathbb{R}\times\overline{\mathcal{O}}}.$$

Therefore,

$$\langle \mathcal{S}_{\mathcal{O}^c\mathcal{O}} F^*, G \rangle_{L^2(\mathbb{R}\times\partial X)} = 2\langle h, \psi \rangle_{L^2(X)},$$
$$\langle \mathcal{S}_{\mathcal{O}\mathcal{O}^c} G^*, F \rangle_{L^2(\mathbb{R}\times\partial X)} = 2\langle \psi, h \rangle_{L^2(X)},$$

and so

$$\langle \mathcal{S}_{\mathcal{O}^c\mathcal{O}} F^*, G \rangle_{L^2(\mathbb{R}\times\partial X)} = \overline{\langle \mathcal{S}_{\mathcal{O}\mathcal{O}^c} G^*, F \rangle}_{L^2(\mathbb{R}\times\partial X)} = \langle F, \mathcal{S}_{\mathcal{O}\mathcal{O}^c} G^* \rangle_{L^2(\mathbb{R}\times\partial X)},$$

and since F and G are arbitrary, this proves our first claim.

Let $G(s,y)=\mathcal{R}_+(\varphi,\psi)(s,y)$ be supported in $\mathbb{R}\times\overline{\mathcal{O}}^c$ and let $F=\frac{1}{2}\mathcal{S}_{\mathcal{O}\mathcal{O}^c}G^*=\mathcal{R}_+(0,\psi)|_{\mathbb{R}\times\overline{\mathcal{O}}}$. We apply Lemma 2.9 to the operator $T_{\mathcal{O}}:L^2_{\mathrm{ac}}(X)\longrightarrow\dot{H}^1_{\mathrm{ac}}(X)$, and let ψ be as in (2.5). Therefore

$$F=\mathcal{R}_+(0,\psi)|_{\mathbb{R}\times\overline{\mathcal{O}}}=\mathcal{R}_+(T_{\mathcal{O}}v,v)\text{ and }\mathcal{R}_+(0,\psi)|_{\mathbb{R}\times\overline{\mathcal{O}}^c}=\mathcal{R}_+(-T_{\mathcal{O}}v,T^*_{\mathcal{O}}T_{\mathcal{O}}v).\tag{2.6}$$

We will show that $\mathcal{S}_{\mathcal{O}^c\mathcal{O}}$ and $\mathcal{S}_{\mathcal{O}\mathcal{O}^c}$ determine $\mathcal{R}_+(0,\psi)|_{\mathbb{R}\times\overline{\mathcal{O}^c}}=\mathcal{R}_+(-T_{\mathcal{O}}v,T^*_{\mathcal{O}}T_{\mathcal{O}}v)$ for the class of $G\in L^2_{\mathrm{ac}}(X)$ such that $G=\mathcal{R}_+(\varphi,\psi)\in C^\infty(\mathbb{R}\times\overline{\mathcal{O}}^c)\cap L^2(\mathbb{R}\times\overline{\mathcal{O}}^c)$, with $\widehat{G}(\lambda,y)\in C^\infty(\partial X)$, for every $\lambda\in\mathbb{R}$, where $\widehat{G}(\lambda,y)$ denotes the Fourier transform of $G(s,y)$ in the variable s. This family includes $G\in C_0^\infty(\mathbb{R}\times\mathcal{O}^c)$ and therefore is dense in $L^2(\mathbb{R}\times\overline{\mathcal{O}}^c)$. This would show that given G in this class, $\mathcal{S}_{\mathcal{O}\mathcal{O}^c}$ and $\mathcal{S}_{\mathcal{O}^c\mathcal{O}}$ determine $\mathcal{R}_+(0,\psi)|_{\mathbb{R}\times\overline{\mathcal{O}}^c}=\frac{1}{2}(G+(\mathcal{S}G^*|_{\mathbb{R}\times\overline{\mathcal{O}}^c}))$. Therefore $\mathcal{S}G^*|_{\mathbb{R}\times\overline{\mathcal{O}}^c}=\mathcal{S}_{\mathcal{O}^c\mathcal{O}^c}G$ is determined for this class of G, and by density for all $G\in L^2(\mathbb{R}\times\overline{\mathcal{O}}^c)$. This argument of course also shows that $\mathcal{S}_{\mathcal{O}\mathcal{O}^c}$ determines $\mathcal{S}_{\mathcal{O}\mathcal{O}}$. So from $\mathcal{S}_{\mathcal{O}\mathcal{O}^c}$ and $\mathcal{S}_{\mathcal{O}^c\mathcal{O}}$ we determine $\mathcal{S}_{\mathcal{O}\mathcal{O}}$ and $\mathcal{S}_{\mathcal{O}^c\mathcal{O}^c}$. This would prove the Proposition. Now we need to prove our claim.

Let $G\in C^\infty(\mathbb{R}\times\partial X)\cap L^2(\mathbb{R}\times\overline{\mathcal{O}}^c)$ and suppose that $\widehat{G}(\lambda,y)\in C^\infty(\partial X)$ for each λ. Let $F=\frac{1}{2}\mathcal{S}_{\mathcal{O}\mathcal{O}^c}G^*$. We want to find the maximum of the quantity

$$\left|\langle\frac{1}{2}\mathcal{S}_{\mathcal{O}^c\mathcal{O}}F^*,H\rangle_{L^2(\mathbb{R}\times\overline{\mathcal{O}}^c)}\right|\tag{2.7}$$

for functions $H(s,y)$ that satisfy the following properties:

P.1. $H(s,y)$ is supported in $\mathbb{R}\times\overline{\mathcal{O}}^c$.
P.2. $\widehat{H}(\lambda,y)\in C^\infty(\mathbb{R}\times\overline{\mathcal{O}}^c)$.
P.3. $\widehat{H}(\lambda,y)=-\widehat{F}(\lambda,y)$ for at least one value of λ, and one $y\in\overline{\mathcal{O}}\cap\overline{\mathcal{O}}^c$.

Here we are using properties of the stationary scattering operator, i.e. the scattering matrix which is defined by conjugating $\mathcal{S}$ with the Fourier transform in the s-variable $\mathcal{A}=\mathcal{F}\mathcal{S}\mathcal{F}^{-1}f$, where $\mathcal{F}$ denotes the Fourier transform in the variable s. For fixed λ, $\mathcal{A}$ is a pseudodifferential operator [28], and since $\widehat{G}(\lambda)$ is C^∞, therefore so is $\widehat{F}$. So property P.3. makes sense.

We claim that the maximum of (2.7) in the class of functions H satisfying P.1, P.2 and P.3 is achieved for the function $H_0(s,y)=-\mathcal{R}_+(0,\psi)|_{\mathbb{R}\times\mathcal{O}^c}$.

As above, let $v\in L^2_{\mathrm{ac}}(X)$ be such that $v=(I+T^*_{\mathcal{O}}T_{\mathcal{O}})\psi$, and so in view of (2.6) $F=\mathcal{R}_+(T_{\mathcal{O}}v,v)$. If $H=\mathcal{R}_+(f,h)\in L^2(\mathbb{R}\times\overline{\mathcal{O}}^c)$, then

$$\langle\frac{1}{2}\mathcal{S}_{\mathcal{O}^c\mathcal{O}}F^*,H\rangle_{L^2(\mathbb{R}\times\mathcal{O}^c)}=\langle\mathcal{R}_+(-T_{\mathcal{O}}v,0),\mathcal{R}_+(f,h)\rangle_{L^2(\mathbb{R}\times\mathcal{O}^c)}=-\langle T_{\mathcal{O}}v,f\rangle_{H^1(X)}.$$

Therefore, by the Cauchy-Schwartz Theorem,

$$|\langle\frac{1}{2}\mathcal{S}_{\mathcal{O}^c\mathcal{O}}F^*,H\rangle_{L^2(\mathbb{R}\times\mathcal{O}^c)}|\leq||T_{\mathcal{O}}v||_{\dot{H}^1_{\mathrm{ac}}(X)}||f||_{\dot{H}^1_{\mathrm{ac}}(X)},$$

and the equality occurs if and only if $f=\kappa T_{\mathcal{O}}v$, $\kappa\neq0$. In this case, since $T^*_{\mathcal{O}}=-T^{-1}_{\mathcal{O}^c}$, it follows that $h=T^{-1}_{\mathcal{O}^c}f=-\kappa T^*_{\mathcal{O}}T_{\mathcal{O}}v$ and hence

$$H=\mathcal{R}_+(f,h)=\kappa\mathcal{R}_+(T_{\mathcal{O}}v,-T^*_{\mathcal{O}}T_{\mathcal{O}}v),$$

We deduce from (2.6) that

$$H=-\kappa\mathcal{R}_+(-T_{\mathcal{O}}v,T^*_{\mathcal{O}}T_{\mathcal{O}}v)=-\kappa\mathcal{R}_+(0,\psi)|_{\mathbb{R}\times\overline{\mathcal{O}}^c}.$$

In view of property P.3, $\kappa=1$. This proves that the maximum of (2.7) for a fixed $G\in C^\infty(\mathbb{R}\times\partial X)\cap L^2(\mathbb{R}\times\mathcal{O}^c)$, $\widehat{G}(\lambda,y)\in C^\infty(\partial X)$ and for H in the class of functions satisfying P.1, P.2 and P.3, is achieved if and only if $H=-\mathcal{R}_+(0,\psi)|_{\mathbb{R}\times\overline{\mathcal{O}}^c}$. This uniquely determines $\mathcal{R}_+(0,\psi)|_{\mathbb{R}\times\overline{\mathcal{O}}^c}$, proves our second claim and ends the proof of the Proposition. $\square$

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Departamento de Matemática, Universidade Federal de Santa Catarina
Campus Universitário Trindade
Florianópolis, SC, Brazil 88040-900
E-mail address: rhora@mtm.ufsc.br

Department of Mathematics, Purdue University
150 North University Street,
West Lafayette Indiana, 47907, USA
E-mail address: sabarre@purdue.edu