EXISTENCE OF RESONANCES IN EVEN DIMENSIONAL POTENTIAL SCATTERING

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1. Introduction and Statement of Results

In this article we show the existence of resonances for the perturbation of the Laplacian in $\mathbb{R}^n$, $n \geq 4$ even, by *any* non-zero real valued potential $V \in C_0^\infty(\mathbb{R}^n)$. The resonances are defined as the poles of the meromorphic continuation of the resolvent or the scattering matrix.

As far as we know this is the first general result about the existence of resonances in even dimensions. We remark that there has been a lot of work on existence of resonances in the semi-classical limit. For the semi-classical Schrödinger operator, the recent result of Sjöstrand [20] implies that for h small enough, $h^{-2}V(x)$ has resonances the number of which grows at least as h^{-n}.

The main result of this article is:

Theorem 1.1. *Let* $V \in C_0^\infty(\mathbb{R}^n;\mathbb{R})$, $n \geq 4$ *even, and* $P = -\Delta + V$. *If* $V \not\equiv 0$ *then the meromorphic continuation of* $(P-\lambda^2)^{-1} : L^2_{\text{comp}}(\mathbb{R}^n) \longrightarrow H^2_{\text{loc}}(\mathbb{R}^n)$, $\Im\lambda > 0$, $\lambda^2 \notin \sigma(P)$, *to the logarithmic plane* Λ *has at least one pole.*

We observe that according to Theorem 1 of [19], the scattering matrix $S(\lambda)$ satisfies

$$S(\overline{\lambda})^* = 2I - S\left(e^{i\pi}\lambda\right), \quad \text{where for} \quad \lambda = |\lambda|e^{i\arg(\lambda)}, \overline{\lambda} = |\lambda|e^{-i\arg(\lambda)},$$

Thus if λ is a pole of S, so is $\overline{\exp(-i\pi)\lambda} = |\lambda|\exp\left(-i(\arg(\lambda)-\pi)\right)$. Unless $\arg(\lambda) = \pi/2$, $\lambda \neq \overline{\lambda\exp(-i\pi)}$. This is the analogue of the fact that in odd dimensions the poles are symmetric with respect to the imaginary axis. Poles that satisfy $\arg(\lambda) = \pi/2$ are square roots of negative eigenvalues.

As in [17] and [18] the method we use is not constructive and does not provide any lower bound on the number of resonances. And unlike [17] and [18], it does not even guarantee the existence of infinitely many resonances. Thus the result is far from the optimal upper bounds obtained by Vodev [26], [27]. If $N(r,a)$ denotes the number of resonances

$$N(r,a) = \#\{\lambda_j \in \Lambda : 0 < |\lambda_j| \leq r, \quad |\arg\lambda_j| \leq a\}, \quad r, a > 1,$$

then for $V \in C_0^\infty(\mathbb{R}^n)$, $n \geq 2$,

$$N(r,a) \leq Ca\left(r^n + (\log a)^n\right).$$

An upper bound of the type Cr^{n+1}, for a different counting function and for $n \geq 4$, had been previously obtained by Intissar in [7].

The sharp upper bound for the odd dimensional scattering by a potential was obtained by Zworski [30]. Also in odd dimensions and for radial potentials a lower bound was obtained in [29]. For an account of the development in pole counting we refer the reader to the survey by Zworski [31], see also [28].

2. Remarks in the Odd Dimensional Case and the Method of Proof

In odd dimensions and in the non-semiclassical case, for a class of non-negative compactly supported potentials in $\mathbb{R}^3$, the existence of resonances was first proved by Lax and Phillips [10]. They actually showed that the number of poles on the imaginary axis contained in a circle of radius r is greater than or equal to Cr^{n-1}. More recently Vasy [25] has extended their result for potentials with a fixed sign. The existence of resonances, without any lower bound, was extended by Menzala and Schonbek [14] to include compactly supported potentials in $\mathbb{R}^n$, n odd, with small negative part in an appropriate sense. Melrose [13] proved that *any* non-zero potential in $\mathcal{C}_c^\infty(\mathbb{R}^3;\mathbb{R})$ has infinitely many resonances and in [17] Zworski and the first author used his observation to show the existence of infinitely many resonances for any elliptic self-adjoint super-exponentially decaying perturbation of the Laplacian in $\mathbb{R}^3$. In [1] Bañuelos and the first author showed the existence of resonances for super-exponentially decaying non-negative potentials in odd dimensions $3 \leq n \leq 9$, and for super-exponentially decaying potentials with non-negative Fourier transforms in all odd dimensions greater than one. Then Zworski and the first author [18] showed the existence of infinitely many resonances for *any* super-exponentially decaying potentials in odd dimensions.

The methods of [18] were extended in [16] to show the existence of resonances for *any* super-exponentially decaying metric perturbation in dimension 5, and for the particular case of conformal perturbations in arbitrary odd dimensions greater than or equal to 3.

The proof of the result of [18] was based on two main arguments. First they used the trace formula, which holds in odd dimensions only,

$$T(t) = \text{ Tr } (U(t) - U_0(t)) = \sum_{\text{resonances}} e^{-i\lambda_j|t|}, \ \ |t| \neq 0, \tag{2.1}$$

in the sense of distributions, where $U(t)$ denotes the perturbed wave group, $U_0(t)$ the free one and where we include square roots of eigenvalues among resonances. This formula was first proved by Bardos, Guillot and Ralston [2] for $|t|$ relatively large. Melrose [12] proved it for $|t| > 0$. This formula has been extended to more general situations in [20] and [22]. In [17] it was extended to elliptic self-adjoint super-exponentially decaying perturbations.

We can see from (2.1) that if there are no resonances $T(t)$ is supported at $\{0\}$. On the other hand, for n odd, the asymptotic expansion of $T(t)$ for t small is of the form

$$T(t) \sim \sum_{j=1}^{\frac{n-1}{2}} d_j(V)\delta^{(n-1-2j)} + \sum_{j\geq\frac{n+1}{2}} d_j(V)|t|^{(2j-n+1)} \tag{2.2}$$

Thus the second sum is zero and (2.2) is an equality. This implies that the regularized heat trace is of the form

$$\text{Tr } (H(t) - H_0(t)) = t^{-\frac{n}{2}} \sum_{j=1}^{\frac{n-1}{2}} C_j(V)t^j, \tag{2.3}$$

where $C_j(V) = \alpha_j d_j(V)$, $\alpha_j \neq 0$.

For $n = 3$ this gives that $C_2(V) = 0$, but $C_2(V)$ is a non-zero multiple of the integral of the square of the potential. Hence the potential must be zero. When $n \geq 5$, since zero is not a resonance, they showed that if there are no negative eigenvalues the rate of decay of the regularized heat trace, as $t \to \infty$, is given by

$$|\text{ Tr } (H(t) - H_0(t))| \leq C(V)t^{-\frac{n}{2}+1}, \text{ for all } t > 0, \tag{2.4}$$

where $H(t)$ and $H_0(t)$ are respectively the perturbed and free heat semigroups. This is in fact an easy consequence of a result of Varopoulos [24], see also Theorem 2.4.2 in [4]. Thus it follows that if there are no resonances, $C_j(V) = 0$, $j \geq 2$. It was observed by Melrose [13] that the existence of infinitely many resonances follows from (2.1) and (2.2).

Recently Christiansen [3] has shown that, in odd dimensions, the the counting function $N(r)$ satisfies

$$\limsup_{r\to\infty} \frac{N(r)}{r(\log r)^{-p}} = \infty, \ \ \forall\, p > 1.$$

In the even dimensional case the scattering matrix is meromorphic on the logarithmic plane Λ and, as mentioned above, (2.1) no longer holds. We use some results of [7] and the methods of [28] to obtain an upper bound on the determinant of the scattering matrix, provided that there are no resonances. This gives an upper bound on the continuation of the scattering phase to Λ. Then we combine this upper bound with the asymptotic expansion of the scattering phase of [15] to show that if there are no poles this expansion has only finitely many terms. We observe that in the odd dimensional case this conclusion can be reached from the bounds on the determinant of the scattering matrix obtained in [28]. This is essentially equivalent to using (2.1), since the bounds obtained in [28] were used to give a new proof of (2.1).

As pointed out in [28], instead of analyzing the behavior of the heat trace as $t \to \infty$, it is more convenient to use the behavior of the derivative of the scattering phase at $\lambda = 0$ to conclude that $C_2(V) = 0$ and hence $V = 0$.

3. PROOF OF THEOREM 1.1

In the even dimensional case the scattering matrix $S(\lambda)$ is a meromorphic function defined on the logarithmic plane Λ. In section 4 we use some of the results of [7] and the methods of [28] to prove

Proposition 3.1. *Let $S_V(\lambda)$, $\lambda \in \Lambda$, be the scattering matrix for $-\Delta + V$, $V \in C_0^\infty(\mathbb{R}^n)$, real valued and $n \geq 4$ even. If $S_V(\lambda)$ is holomorphic in Λ, then, for $\lambda = e^z$, $z \in \mathbb{C}$,*

$$|\det S_V(e^z)| \leq C \exp(\exp C|z|). \tag{3.1}$$

We postpone the proof of Proposition 3.1 until the next section. We will use it to prove Theorem 1.1.

Proof. The proof is by contradiction. Suppose that $S(\lambda)$ is holomorphic. Theorem 1 of [19] states that

$$S(\lambda)\left(2I - S\left(e^{i\pi}\lambda\right)\right) = I, \quad \lambda \in \Lambda.$$

Hence $\det S(\lambda)$ never vanishes and we deduce that there exists a holomorphic function $\sigma(\lambda)$, $\lambda \in \Lambda$, such that

$$\det S(\lambda) = \exp(\sigma(\lambda))$$

It follows from the Borel-Carathéodory theorem, see Theorem 5.5 of [23], and (3.1) that

$$|\sigma(e^z)| \leq K \exp(2C|z|), \tag{3.2}$$

where here and throughout this proof K denotes some constant. But we know that for $\lambda \in \mathbb{R}^+$, $\frac{1}{2\pi i}\sigma(\lambda)$ is the scattering phase and that in even dimensions it is given by, see for example [15],

$$\sigma(\lambda) = \sum_{j=1}^{\frac{n}{2}-1} \alpha_j(V)\lambda^{n-2j} + O\left(\lambda^{-\infty}\right), \quad \lambda \to \infty \tag{3.3}$$

We remark that the fact that the error in (3.3) is $O(\lambda^{-\infty})$ is due to the fact that the trace of the wave group in even dimenions has an expansion near $t = 0$ in even powers of t with only finitely many singular terms, see for example Theorem 17.5.5 of [6].

Now set

$$F(z) = \sigma(e^z) - \sum_{j=1}^{\frac{n}{2}-1} \alpha_j(V) \exp\left((n-2j)z\right) \tag{3.4}$$

and now we work with the z variable on the complex plane $\mathbb{C}$. For C in (3.2), let $M = \max\{2C, n-2\}$. Thus we find that

$$|F(z)| \leq K \exp(M|z|) \tag{3.5}$$

Now let $H(z) = F(z)\exp\left((M+1)z\right)$. It follows from (3.5) that

$$|H(z)| \le K\exp\left((2M+1)|z|\right). \tag{3.6}$$

We deduce from (3.3) that

$$|H(z)| \le K\exp(-z), \quad \Im z = 0, \quad z \to \infty \tag{3.7}$$

On the other hand it follows from (3.5) that

$$|H(z)| \le K\exp(z), \quad \Im z = 0, \quad z \to -\infty \tag{3.8}$$

Now let $b > 0$ and set $H_1(z) = H(z)e^{-ibz}$. Then $H_1(z)$ is of exponential type and, for $\Im z = 0$ we have $|H_1(z)| \le K\exp(-|z|)$, and for $\Re z = 0$, $|H_1(z)| \le K\exp((2M+1+b)|z|)$. It follows from the Phragmén-Lindelöf principle that, there exists $C > 0$, independent of b for which, by writing $z = re^{i\theta}$, $|H_1(re^{i\theta})| < Ce^{((2b+M+1)\sin\theta - \cos\theta)r}$, for $0 \le \theta \le \frac{\pi}{2}$. Let $\theta_0 \in (0, \frac{\pi}{2})$ be such that $\tan\theta_0 = \frac{1}{b+M+1}$. Applying Phrägmen-Lindelöf to the region , $= \{z : 0 \le \theta \le \theta_0\}$ we have that $|H_1(z)| \le C$ for $z \in$, . A similar argument can be applied to the show that $H_1(z)$ is bounded in region , $_1 = \{z : \pi - \theta_0 < \theta < \pi\}$. Finally we apply Phrägmen-Lindelöf to the region $\{z : \theta_0 < \theta < \pi - \theta_0\}$ to deduce that H_1 is bounded on the upper half-plane. Thus we find that $|H(z)| < C\exp(-b\Im z)$ for $\Im z \ge 0$ Since b is arbitrary, we find that $H(z) = 0$ and therefore $F(z) = 0$. This is a particular case of an argument due to Carlson, see Theorem 5.8 of [23].

In particular we conclude that

$$\sigma(\lambda) = \sum_{j=1}^{\frac{n}{2}-1} \alpha_j(V)\lambda^{n-2j}, \qquad \lambda > 0. \tag{3.9}$$

We also know by the behavior of the scattering phase at $\lambda = 0$, see for example section 3 of [32],

$$\sigma'(\lambda) = \lambda^{n-3} f(\lambda, \lambda^{n-2}\log(\lambda)), \quad f \in C^\infty, \quad \lambda > 0. \tag{3.10}$$

It follows from (3.9) and (3.10) that $\alpha_2(V) = 0$. But it is well known that $\alpha_2(V) = C\int V^2(x)dx$, $C \ne 0$. Hence $V = 0$. This ends the proof of Theorem 1.1. □

4. Upper Bounds on the Determinant of the Scattering Matrix

Now we prove Proposition 3.1.

Proof. We follow the proof of Proposition 6 of [28]. Let $R_0(\lambda)$ denote the holomorphic continuation of $\left(-\Delta + \lambda^2\right)^{-1}$ to Λ. We recall from equation (4.5) of [28], see also equation (3.13) of [7], that

$$S_V(\lambda) = I + A_V(\lambda)$$

where

$$A_V(\lambda) = E^\rho(-\lambda)V\left(I - H_V(\lambda)\right)^{-1}{}^tE^\rho(\lambda),$$
$$H_V(\lambda) = \rho R_0(\lambda)V, \quad \rho \in C_0^\infty, \quad \rho V = V$$

and $E^\rho(\lambda)$ has Schwartz kernel given by

$$C_n\lambda^{\frac{n-2}{2}}e^{i\lambda\langle x,\omega\rangle}\rho(x).$$

Let $\mu_j(A(\lambda))$ denote the characteristic values of $A_V(\lambda)$, then

$$|\det S_V(\lambda)| \le \prod_{j=1}^{\infty}\left(1 + \mu_j(A_V(\lambda))\right). \tag{4.1}$$

To estimate $\mu_j(A(\lambda))$ we use that

$$\mu_j(A(\lambda)) \leq \mu_j(E^\rho(-\lambda))||V||.||\left(I - H_V(\lambda)\right)^{-1}||.||^t E^\rho(\lambda)||. \tag{4.2}$$

Setting $\lambda = e^z$ we obtain from the argument used in [28], see also the proof of Corollary 1.1 of [7], and Proposition 2 of [30], that

$$\mu_j(E^\rho(-e^z)) \leq C\exp(\exp C|z| - j^{\frac{1}{n-1}}/C), \quad C > 0. \tag{4.3}$$

It is easy to see that

$$||^t E^\rho(e^z)|| \leq C\exp(\exp C|z|). \tag{4.4}$$

To estimate $||\left(I - H_V(e^z)\right)^{-1}||$ we use, as in [28], Theorem V.5.1 of [5]. It follows that

$$||\left(I - H_V(e^z)\right)^{-1}|| \leq$$
$$|\det\left(I + (H_V(e^z))^{\frac{n}{2}+1}\right)|^{-1}.\det\left(I + |H_V(e^z)|^{\frac{n}{2}+1}\right)$$

From Proposition 2.1 of [7] we obtain

$$\begin{gathered}|\det\left(I + (H_V(e^z))^{\frac{n}{2}+1}\right)| \leq \\ \det\left(I + |H_V(e^z)|^{\frac{n}{2}+1}\right) \leq \exp(C\exp(n+1)|z|).\end{gathered} \tag{4.5}$$

Theorem I.11 of [11] states that if $f(z)$ is a holomorphic function in the disk $|z| \leq 2eR$, with $f(0) = 1$, and $\eta \in (0, \frac{3e}{2})$, then outside a family of disks the sum of whose radii is not greater than $4\eta R$, we have for $|z| \leq R$

$$\ln|f(z)| > -\left(2 + \ln\frac{3e}{2\eta}\right)\ln M(2eR),$$
$$M(s) = \sup\{|f(z)|; |z| \leq s\}.$$

Then for $\eta = \frac{1}{20}$ there exists $\rho \in (\frac{R}{2}, R)$ such that the boundary of the disk of center zero and radius ρ does not intersect any of the excluded disks. Otherwise the sum of their radii would have to be greater than or equal to $R/4$. Since $R < 2\rho$ we obtain, for $m(\rho) = \inf\{|f(z)| : |z| = \rho\}$,

$$\ln m(\rho) > -(2 + \ln(30e))\ln M(4e\rho). \tag{4.6}$$

If $\det\left(I + (H_V(e^z))^{n/2+1}\right) = az^m f(z)$ where $f(z)$ is entire and $f(0) = 1$. Then it follows from (4.5) and (4.6) that for every $R > 0$, there exists $\rho \in (R/2, R)$ such that

$$\begin{gathered}|\det\left(I + (H_V(e^z))^{\frac{n}{2}+1}\right)| \geq K(m)\exp(-C\exp(4e(n+1)|z|)), \\ |z| = \rho\end{gathered} \tag{4.7}$$

Hence it follows from (4.5) and (4.7) that, for these values of ρ,

$$||\left(I - H_V(e^z)\right)^{-1}|| \leq K(m)\exp(2C\exp(4(n+1)e|z|)), \quad |z| = \rho. \tag{4.8}$$

Since $S_V(e^z)$ is holomorphic, so is $\left(I - H_V(e^z)\right)^{-1}$, see [19]. Thus given any $z \in \mathbb{C}$, there exist $\rho \in (|z|, 2|z|)$ and $\rho_1 \in (|z|/2, |z|)$ such that (4.8) holds for the disks with radii ρ and ρ_1 respectively. It follows from Hadamard's three circles theorem that

$$||\left(I - H_V(e^z)\right)^{-1}|| \leq K(m)\exp(2C\exp(8(n+1)e|z|)). \tag{4.9}$$

It follows from (4.2), (4.3), (4.4) and (4.9) that there exists $C > 0$ such that

$$\mu_j\left(A_V(e^z)\right) \leq C\exp(\exp C|z| - \frac{1}{C}j^{\frac{1}{n-1}}). \tag{4.10}$$

Then, as in [28], see also [17], (3.1) follows from (4.1) and (4.10). ☐

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