1. Introduction

These notes are an introduction to Lebesgue integral. The reference is Introduction to Hilbert Spaces with Applications by Lokenath Debnath and Piotr Mikusiński. Second Edition, Academic press 1998

2. Lecture I: Step Funtions

The characteristic function of a set $X \subset \mathbb{R}$ is defined

$$\chi_X(x) = \begin{cases} 1 & \text{if} \quad x \in X \\ \\ 0 & \text{if} \quad x \notin X \end{cases}$$

A step function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function of the form

$$f(x) = \sum_{j=1}^{N} c_j \chi_{[a_j, b_j)}. \tag{2.1}$$

The representation (2.1) is not unique. Example

$$\chi_{[1,2)} = \chi_{[1,\frac{3}{2})} + \chi_{[\frac{3}{2},2)}.$$

Proposition 2.1. *If the cubes Q_j are disjoint, and we use the minimum number of cubes, then* (2.1) *is unique and is called the basic representation of f.*

Proof. Let $\{a_1, a_2, ..., a_M\}$ be the set of discontinuities of f. These are the points where the graph of f has a jump. We can order them as $a_1 < a_2 < ... < a_M$. Then

$$f(x) = \sum_{k=2}^{M} \alpha_k \chi_{[a_{k-1}, a_k)}, \quad \alpha_k = f(a_k). \tag{2.2}$$

□

We will need the following

Lemma 2.1. *If f and g are step functions, then $f + g$, λf, $\lambda \in \mathbb{R}$, $|f|$, $\max(f, g)$, $\min(f, g)$ are step functions.*

Proof. The only item which is perhaps non-trivial is that $|f|$ is a step function. For that we use the basic representation of f to conlcude that

$$f(x) = \sum_{k=2}^{M} \alpha_k \chi_{[a_{k-1}, a_k)} \Longrightarrow |f(x)| = \sum_{k=2}^{M} |\alpha_k| \chi_{[a_{k-1}, a_k)}.$$

The other important thing is to realize that (prove this as an exrecise)

$$\max(f, g) = \frac{1}{2}(f + g + |f - g|), \quad \min(f, g) = \frac{1}{2}(f + g - |f - g|).$$

□

Definition 2.1. *The (Lebesgue) integral of a step function*

$$f(x) = \sum_{j=1}^{N} c_j \chi_{[a-j, b_j)}$$

is

$$\int f \, dx = \sum_{j=1}^{N} c_j (b_j - a_j).$$

Proposition 2.2. *If f is a step function, $\int f\ dx$ is independent of its representation.*

This is an easy consequence of the fact that $\int f\ dx$ is the Riemann integral of the function.

Proposition 2.3. *For any step functions f and g we have*

$$(1) \qquad \int (f+g)\ dx = \int f\ dx + \int g\ dx$$

$$(2) \qquad \int \lambda f\ dx = \lambda \int f\ dx, \quad \lambda \in \mathbb{R}$$

$$(3) \qquad \text{if } f \leq g \Longrightarrow \int f\ dx \leq \int g\ dx$$

$$(4) \qquad \left| \int f\ dx \right| \leq \int |f|\ dx$$

Proof. (1) and (2) are left as exercise. To prove (3), in view of (2) , we only need to show that if $f \geq 0$ then $\int f\ dx \geq 0$. In this case one only needs to write the basic representation of f (2.2) and notice that if $f \geq 0$, then $\alpha_k \geq 0$. Then $\int f\ dx = \sum_{k=2}^{M} \alpha_k (a_k - a_{k-1}) \geq 0$.

To prove (4) just observe that $-|f| \leq f \leq |f|$, then from (3)

$$\int -|f|\ dx \leq \int f\ dx \leq \int |f|\ dx.$$

□

3. Lecture II: The Lebesgue Integral

Next we prove

Lemma 3.1. *Let $a, b \in \mathbb{R}$, $a < b$ and let $[a_1, b_1), [a-2, b_2), ...$ be subintervals of $[a, b)$ satisfying*

$$(1) \qquad [a_j, b_j) \cap [a_k, b_k) = \emptyset \quad \text{if} \quad j \neq k$$

$$(2) \qquad [a,b) = \bigcup_{j=1}^{\infty} [a_j, b_j) = [a, b)$$

Then

$$\sum_{j=1}^{\infty} (b_j - a_j) = b - a.$$

Proof. (This proof is due to Bartlomiej Siudeja) Since $[a_j, b_j) \subset [a, b)$ and (2) holds we have for any $N \in \mathbb{N}$,

$$\sum_{j=1}^{N} b_j - a_j \leq b - a.$$

Thus

$$\sum_{j=1}^{\infty} b_j - a_j \leq b - a.$$

Suppose that there exists $L > 1$ such that

$$\sum_{j=1}^{\infty} b_j - a_j = \frac{b-a}{L} \tag{3.1}$$

Fix any interval $[a_{j_0}, b_{j_0})$. The linear transformation

$$T_{j_0}(x) = \frac{b_{j_0} - a_{j_0}}{b - a} x + \frac{b a_{j_0} - b_{j_0} a}{b - a}$$

maps $[a, b)$ into $[a_{j_0}, b_{j_0})$. Therefore $T_{j_0}([a_j, b_j))$ gives a cover $[T_{j_0}(a_j), T_{j_0}(b_j))$, $j = 1, 2, ...,$ of $[a_{j_0}, b_{j_0})$. Moreover, since the length of $[T_{j_0}(a_j), T_{j_0}(b_j))$ is equal to $T_{j_0}(b_j) - T_{j_0}(a_j) = \frac{b_{j_0} - a_{j_0}}{b-a}(b_j - a - j)$ we find that

$$\sum_{j=1}^{\infty} (T_{j_0}(b_j) - T_{j_0}(a_j)) = \frac{b_{j_0} - a_{j_0}}{L}.$$

This gives a cover of the interval $[a, b)$ formed by the *disjoint* intervals

$$T_m([a_j, b_j)), \quad j = 1, 2, ...; m = 1, 2, ...$$

with the property that

$$\sum_{m,j=1}^{\infty} (T_m(b_j) - T_m(a_j)) = \frac{b - a}{L^2}$$

Given $\epsilon > 0$, let N be such that $\frac{b-a}{L^N} < \epsilon$. By repeating this argument N times we find that there exist disjoint intervals $[c_j, d_j)$, $j = 1, 2, ...$ such that

$$[a, b) \subset \bigcup_{j=1}^{\infty} [c_j, d_j), \qquad \sum_{j=1}^{\infty} (d_j - c_j) < \epsilon.$$

Then it follows that

$$[\frac{b + 2a}{3}, \frac{a + 2b}{3}] \subset \bigcup_{j=1}^{\infty} (c_j - \frac{\epsilon}{2^j}, d_j + \frac{\epsilon}{2^j})$$

The compactness of $[\frac{b+2a}{3}, \frac{a+2b}{3}]$ implies that there exist $j_0, j_1, ..., j_N$ such that

$$[\frac{b + 2a}{3}, \frac{a + 2b}{3}] \subset \bigcup_{k=1}^{N} (c_{j_k} - \frac{\epsilon}{2^{j_k}}, d_{j_k} + \frac{\epsilon}{2^{j_k}}).$$

Since this is a finite union, we must have

$$\frac{b - a}{3} \leq \sum_{k=1}^{N} \left[d_{j_k} - c_{j_k} + 2\frac{\epsilon}{2^{j_k}} \right] \leq \sum_{j=1}^{\infty} (d_j - c_j) + 2\epsilon \sum_{j=1}^{\infty} 2^{-j} < 3\epsilon.$$

This is an absurd. □

Definition 3.1. *A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is Lebesgue integrable if there exists a sequence of step functions f_n, $n = 1, 2, ...$ such that*

$$\sum_{j=1}^{\infty} \int |f_n| \, dx < \infty \tag{3.2}$$

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \textit{for every } x \quad \textit{such that} \quad \sum_{j=1}^{\infty} |f_n(x)| < \infty. \tag{3.3}$$

The integral of f is then defined by

$$\int f \ \ dx = \sum_{n=1}^{\infty} \int f_n \, dx.$$

If f and f_n, $n = 1, 2, ...$ satisfy (3.2) *and* (3.3) *we say that*

$$f \sim f_1 + f_2 + ... \tag{3.4}$$

Theorem 3.1. *If f_n, and g_n, $n = 1, 2, ...$ are step functions and*

$$f \sim \sum_{n=1}^{\infty} f_n \quad \textit{and} \quad f \sim \sum_{n=1}^{\infty} g_n,$$

$$\int f \ \ dx = \sum_{n=1}^{\infty} \int f_n \ dx = \sum_{n=1}^{\infty} \int g_n \ dx.$$

The proof of Theorem 3.1 will be divided into several Lemmas.

Lemma 3.2. *Let f_n, $n = 1, 2, ...$ be a non-increasing sequence of step functions such that* $\lim_{n\to\infty} f_n(x) = 0$ *for all $x \in \mathbb{R}$, then* $\lim_{n\to\infty} \int f_n \ dx = 0$.

Proof. Since $f_1 \geq f_2 \geq ...$ and $f_n(x) \to 0$, it follows that $f_n(x) \geq 0$ for $n = 1, 2, ...$ and $x \in \mathbb{R}$. Thus the sequence $s_n = \int f_n \ dx$ satisfies

$$s_1 \geq s_n \geq s_{n+1} \geq 0.$$

Thus s_n converges. Let

$$\lim_{n\to\infty} \int f_n \ dx = L.$$

As $f_n(x) \leq f_1(x)$ and f_1 is a step function, it follows that there exists an interval $[a, b)$ such that $f_n(x) = 0$ if $x \notin [a, b)$. We know that f_n has a basic representation, that is

$$f_n = \sum_{j=1}^{N} C_{n_j} \chi_{[a_{n_j}, b_{n_j})},$$

with the intervals $[a_{n_j}, b_{n_j})$ disjoint and N the smallest number of intervals.

Let $\alpha > 0$ and for $n = 1, 2, ...$ let

$$A_n = \{x \in [a, b) : f_n(x) < \alpha\} = \bigcup j[a_{n_j}, b_{n_j}); \quad C_{n_j} < \alpha.$$

Since $f_n(x) \geq f_{n+1}(x)$, it follows that $A_n \subset A_{n+1}$. Since $f_n(x) \to 0$, for all $x \in \mathbb{R}$,

$$[a, b) = \bigcup_{j=1}^{\infty} A_n$$

Let

$$B_1 = A_1, \quad B_n = A_n \setminus A_{n-1}, \quad n = 2, 3, ...$$

The sets B_j satisfy the following properties

$$B_n = \bigcup j = 1^{M(n)} [c_{n,j}, d_{n,j}), \quad [c_{n,j}, d_{n,j}), \quad \text{are disjoint intervals}$$

$$B_i \cap B_j = \emptyset \quad \text{if} \quad i \neq j.$$

$$[a, b) = \bigcup_{j=1}^{\infty} B_n$$

From Lemma 3.1

$$\sum n = 1^{\infty} \sum_{j=1}^{M(n)} (d_{n,j} - c_{n,j}) = b - a.$$

Pick N such that

$$\sum n = N + 1^{\infty} \sum_{j=1}^{M(n)} (d_{n,j} - c_{n,j}) < \alpha. \tag{3.5}$$

Let

$$F = \bigcup_{n=1}^{N} B_n = A_N.$$

Notice that if $x \in F$, then $x \in A_N$ and hence $F_N < \alpha$ if $x \notin F$ $f_N(x) \leq \max f_1$ hence, as f_N is a step function

$$\int f_N \, dx \leq \int_F f_N \, dx + \int_{[a,b)\setminus F} f_N \, dx$$

and in view of (3.5) and the definition of A_N

$$\int f_N \, dx \leq \alpha(b - a + \max f_1)$$

So given $\epsilon > 0$ there exists N so that for all $n \geq N$ $\int f_n \, dx \leq \int f_N \, dx < \epsilon$. This ends the proof. □

Lemma 3.3. *Let g_n and h_n be non-decreasing sequences of step functions. If*

$$\lim_{n\to\infty} h_n(x) \leq \lim_{n\to\infty} g_n(x), \quad \forall \; x \in \mathbb{R},$$

then for any $k \in \mathbb{N}$,

$$\int h_k \, dx \leq \lim_{n\to\infty} \int g_n \, dx.$$

Proof. Fix $k \in \mathbb{N}$ and set

$$F_n = h_k - \min(h_k, g_n),$$

$F_{n+1} \leq F_n$ and $F_n \to 0$. Indeed, just observe that

$$F_{n+1}(x) - F_n(x) = \min(h_k(x), g_n(x)) - \min(h_k(x), g_{n+1}(x))$$

If $\min(h_k(x), g_n(x)) = h_k(x)$, then as $g_{n+1} \geq g_n$, then $\min(h_k(x), g_{n+1}(x)) = h_k(x)$ so in this case $F_{n+1}(x) - F_n(x) = 0$. If $\min(h_k(x), g_n(x)) = g_n(x)$ then $\min(h_k(x), g_{n+1}(x)) \leq g_{n+1}(x)$ and thus, using again that $g_{n+1} \geq g_n$, $F_{n+1}(x) - F_n(x) \leq$.

Since $\lim_{n\to\infty} h_n(x) \leq \lim_{n\to\infty} g_n(x)$ for n large $h_k(x) \leq g_n(x)$ and thus $F_n(x) \to 0$ for all $x \in \mathbb{R}$. Thus from Lemma 3.2

$$0 = \lim_{n\to\infty} \int [h_k - \min(h_k, g_n)] \, dx = \int h_k \, dx - \lim_{n\to\infty} \min(h_k, g_n) \, dx \leq \lim_{n\to\infty} g_n \, dx.$$

. □

Corollary 3.1. *Under the hypotheses of Lemma 3.3,*

$$\lim_{n\to\infty} \int h_n \, dx \leq \lim_{n\to\infty} \int g_n \, dx.$$

If

$$\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} g_n(x), \quad \forall \; x \in \mathbb{R},$$

then

$$\lim_{n\to\infty} \int h_n \, dx = \lim_{n\to\infty} \int g_n \, dx.$$

The proof is left as an exercise. Finally we need

Lemma 3.4. *If* $f \sim f_1 + f_2 + ...$ *and* $f \geq 0$, *then*

$$\int f_1 \, dx + \int f_2 \, dx + ... \geq 0.$$

Proof. Since $\sum_{n=1}^{\infty} \int |f_n| \, dx < \infty$, for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} \int |f_n| \, dx < \epsilon$$

Let

$$g_n = f_1 + f_2 + ... + f_N + |f_{N+1}| + ... + |f_{N+n}| \quad \text{and } h_n = \max(g_n, 0).$$

Then $g_n(x) \geq \sum_{j=1}^{N+n} f_j(x)$. Moreover at the points where $\sum_{n=1}^{\infty} |f_n(x)| < \infty$,

$$f(x) = \sum_{n=1}^{\infty} f_n(x),$$

thus

$$\lim_{n\to\infty} g_n(x) \geq f(x) \quad \text{provided } \sum_{n=1}^{\infty} |f_n(x)| < \infty.$$

From the d efinition of g it is clear that

$$\lim_{n\to\infty} g_n(x) = \infty \quad \text{provided } \sum_{n=1}^{\infty} |f_n(x)| = \infty.$$

Hence

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} h_n(x) \geq 0.$$

From Corollary 3.1

$$\lim_{n\to\infty} \int g_n \, dx = \lim_{n\to\infty} \int h_n \, dx \geq 0.$$

So we have

$$\sum_{n=1}^{N} \int f_n \, dx + \sum_{n=1}^{N} \int |f_n| \, dx \geq 0.$$

Since

$$-\sum_{n=1}^{N} \int |f_n| \, dx \leq \sum_{n=1}^{N} \int f_n \, dx.$$

Thus

$$\sum_{n=1}^{N} \int f_n \, dx - \sum_{n=1}^{N} \int |f_n| \, dx \leq \sum_{n=1}^{\infty} \int f_n \, dx$$

Adding $2\sum_{n=1}^{N} \int |f_n| \, dx$ to both sides of this equation gives

$$0 \leq \sum_{n=1}^{N} \int f_n \, dx + \sum_{n=1}^{N} \int |f_n| \, dx \leq \sum_{n=1}^{\infty} \int f_n \, dx + 2\sum_{n=1}^{N} \int |f_n| \, dx \leq \sum_{n=1}^{\infty} \int f_n \, dx + 2\epsilon.$$

So

$$\sum_{n=1}^{\infty} \int f_n \, dx \geq -2\epsilon.$$

□

We are finally ready to prove Theorem 3.1.

Proof. Since $f \sim \sum_{n=1}^{\infty} f_n$ and $f \sim \sum_{n=1}^{\infty} g_n$ thus $0 \sim \sum_{n=1}^{\infty} f_n - g_n$ and $0 \sim \sum_{n=1}^{\infty} g_n - f_n$. From Lemma 3.4 we deduce that

$$0 \leq \sum_{n=1}^{\infty} \int (f_n - g_n) \, dx = \sum_{n=1}^{\infty} \int f_n \, dx - \sum_{n=1}^{\infty} \int g_n \, dx$$

$$0 \leq \sum_{n=1}^{\infty} \int (g_n - f_n) \, dx = \sum_{n=1}^{\infty} \int g_n \, dx - \sum_{n=1}^{\infty} \int f_n \, dx$$

This ends the proof of the Theorem. □

We should mention the following consequence of Theorem 3.1 and Lemma 3.4

Corollary 3.2. *If f is integrable and $f \geq 0$ then $\int f \, dx \geq 0$.*

Definition 3.2. *The space of Lebesgue integrable functions is denoted by $L^1(\mathbb{R})$.*

Theorem 3.2. *The space $L^1(\mathbb{R})$ is a vector space over $\mathbb{R}$.*

The proof is left as an exercise.

Theorem 3.3. *If $f \in L^1(\mathbb{R})$ then $|f| \in L^1(\mathbb{R})$ and $\left|\int f\right| \leq \int |f|$.*

Proof. Let f_n, $n = 1, 2, ...$ be step functions and let $f \sim f_1 + f_2 + ...$ Let

$$Z = \{x \in \mathbb{R} : \sum |f(x)| < \infty\}$$

$$S_n(x) = \sum_{j=1}^{n} f_j(x).$$

We have

$$f(x) = \lim S_n(x), \quad \forall \;\; x \in Z$$

Thus

$$|f(x)| = \lim |S_n(x)|, \quad \forall \;\; x \in Z$$

Let $g_1 = |f_1|$, $g_2 = |f_1 + f_2| - |f_1|$, $g_n = |S_n| - |S_{n-1}|$. Thus

$$|g_n| = ||S_n| - |S_{n-1}|| \leq |S_n - S_{n-1}| = |f_n|$$

As a consequence we have

$$\sum \int |g_n| \, dx < \infty. \tag{3.6}$$

Let

$$u(x) = \sum_{n=1}^{\infty} |f_n(x)|, \quad x \in Z$$

$$u(x) = 0 \;\; x \notin Z.$$

Then $u \sim |f|1 + |f_2| + ...$ and hence $u \in L^1(\mathbb{R})$. We claim that

$$|f(x)| + u(x) \sim \sum_{n=1}^{\infty} g_n + |f_n|$$

It follows from (3.6) that $\sum_{n=1}^{\infty} \int |g_n + |f_n|| \ dx < \infty$. Notice that

$$\sum_{n=1}^{N} g_n + |f_n(x)| = |f_1 + ... + f_N| + \sum_{n=1}^{N} |f_n|$$

Thus

$$\sum_{n=1}^{\infty} g_n(x) + |f_n(x)| < \infty \quad \text{if and only if } x \in Z.$$

Moreover, for

$$\text{if} \quad x \in Z \quad \text{then} \quad |f(x)| + u(x) = \sum_{n=1}^{\infty} g_n(x) + |f_n(x)|.$$

So we conlcude that $u + |f| \in L^1(\mathbb{R})$ and hence $|f| \in L^1(\mathbb{R})$. □

Corollary 3.3. *If $f \sim f_1 + f_2 + ...$ then $\int |f| \ dx \leq \sum \int |f_n| \ dx$.*

Proof. In the notation of the proof of Theorem 3.3

$$\int (|f| + u) \ dx = \sum_{n=1}^{\infty} \int (g_n + |f_n|) \ dx$$

As $\int u \ dx = \sum_{n=1}^{\infty} \int |f_n| \ dx$ and

$$\sum_{n=1}^{N} \int g_n \ dx = \int \left| \sum_{n=1}^{N} f_n \right| \ dx \leq \sum_{n=1}^{N} \int |f_n| \ dx$$

the result follows. □

Corollary 3.4. *If $f, g \in L^1(\mathbb{R})$, then $\min(f, g)$ and $\max(f, g)$ are also in $L^1(\mathbb{R})$.*

The proof is left as an exercise.
Next we prove

Theorem 3.4. *Let $f_n \in L^1(\mathbb{R})$, $n = 1, 2...$ be a sequence of functions such that*

$$\sum_{n=1}^{\infty} \int |f_n| \ dx < \infty.$$

$$f(x) = \sum_{n=1}^{\infty} |f_n(x)|, \quad \textit{for all} \quad x \in \mathbb{R} \quad \textit{such that} \sum_{n=1}^{\infty} |f_n(x)| < \infty.$$

Then $f \in L^1(\mathbb{R})$ and we say that $f \sim \sum_{n=1}^{\infty} f_n$.

The proof of this is based on the following

Lemma 3.5. *If $f \in L^1(\mathbb{R})$ then for all $\epsilon > 0$ there exists a sequence of step functions such that $f \sim \sum_{n=1}^{\infty} f_n$ and $\sum \int |f_n| \ dx < \int |f| \ dx + \epsilon$.*

Proof. Since $f \in L^1(\mathbb{R})$ there exists a sequence of step functions g_n, $n = 1, 2...$ such that

$$f \sim g_1 + g_2 + ...$$

For any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} \int |g_n| < \frac{\epsilon}{2}.$$

Let $f_1 = \sum_{n=1}^{N} g_n$ and $f_n = g_{N+n-1}$, $n \geq 2$. Then $f - f_1 \sim f_2 + f_3 + ...$ and hence

$$\int |f - f_1| \, dx \leq \sum_{n=2}^{\infty} \int |f_n| \, dx < \frac{\epsilon}{2}.$$

In particular this gives that

$$\int |f_1| \, dx - \int |f| \, dx < \frac{\epsilon}{2}.$$

But then

$$\sum_{n=1}^{\infty} |f_n| \, dx \leq \sum_{n=2}^{\infty} \int |f_n| \;\; dx + \int |f_1| \, dx \leq \int |f| \, dx + \epsilon.$$

□

Now we are ready to prove Theorem 3.4.

Proof. Since $f_n \in L^1(\mathbb{R})$, there exists a sequence $f_{n,j}$, $j = 1, 2, ..$ of step functions such that

$$\sum_{j=1}^{\infty} \int |f_{n,j}| \, dx < \infty$$

$$f_n(x) = \sum j = 1^{\infty} f_{n,j}(x), \;\; \forall \;\; x \quad \text{such that} \quad \sum j = 1^{\infty} |f_{n,j}(x)| < \infty$$

$$\int |f_{n,j}| \, dx \leq \int |f_n| \, dx + 2^{-n}.$$

Notice that if

$$\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |f_{n,j}(x)| < \infty \quad \text{then in particular} \quad \sum_{n=1}^{\infty} |f_n(x)| < \infty.$$

Therefore

$$f(x) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} f_{n,j}(x) \quad \text{for all} \quad x \quad \text{such that} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |f_{n,j}(x)| < \infty.$$

So we conclude that $f \sim \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} f_{n,j}$. This proves the Theorem. □

4. Lecture III. Completeness of $L^1(\mathbb{R})$

Definition 4.1. *We say that two functions $f, g \in L^1(\mathbb{R})$ are equivalent, $f \equiv g$, if $\int |f(x) - g(x)| \, dx = 0$.*

It is very easy to prove that this relation is reflexive, symmetric and transitive, so it is an equivalence relation. We will denote

$$[f] \stackrel{\text{def}}{=} \{g \in L^1(\mathbb{R}) : \int |f - g| \, dx = 0\}$$

and redefine the space

$$L^1(\mathbb{R}) = \{[f] : f \in L^1(\mathbb{R})\}.$$

Theorem 4.1. *The space $L^1(\mathbb{R})$ equipped with the norm $||f|| = \int |f|\, dx$ is a normed vector space.*

Proof. The only issue is whether this is a norm. But from the definition, if $||f|| = 0$ it follows that $f \equiv 0$. □

From now on, as a matter of convenience, we will use f rather than f to identify elements of $L^1(\mathbb{R})$. This is the main result of this section:

Theorem 4.2. *$L^1(\mathbb{R})$ is a complete normed space (i.e. a Banach space).*

The proof is divided into two lemmas:

Lemma 4.1. *Let $f_j \in L^1(\mathbb{R})$, $j = 1, 2, ...$ If $\sum_{j=1}^{\infty} \int |f_j|\, dx < \infty$, then*
i) $Y = \{x : \sum_{j=1}^{\infty} |f_j(x)| = \infty\}$ is a null set
ii) There exists $f \in L^1(\mathbb{R})$ such that

$$f(x) = \sum_{j=1}^{\infty} f_j(x) \ \ \forall \ \ x \text{ such that } \ \ \sum_{j=1}^{\infty} |f_j(x)| < \infty.$$

iii) $\lim_{N\to\infty} ||f - \sum_{j=1}^{\infty} f_j|| = 0$.

Proof. To prove (i) we define

$$g_{2n} = -f_n, \ \ g_{2n-1} = f_n, \ \ n = 1, 2, ...$$

We claim that

$$\chi_Y \sim g_1 + g_2 + ...$$

Indeed, as $\sum_{j=1}^{\infty} \int |f_j|\, dx < \infty$, it follows that $\sum_{j=1}^{\infty} \int |g_j|\, dx < \infty$. And

$$\chi_Y(x) = \sum_{j=1}^{\infty} g_j(x) = 0 \ \ \forall \ \ x \quad \text{such that} \quad \sum_{j=1}^{\infty} |g_j(x)| < \infty.$$

Hence

$$\int \chi_Y \, dx = 0 \Longrightarrow Y \quad \text{is null} .$$

This proves (i). To prove (ii), let

$$f(x) = \begin{cases} \sum_{j=1}^{\infty} f_j(x) & \text{if} \quad \sum_{j=1}^{\infty} |f_j(x)| < \infty \\ \\ 0 \quad \text{if} \quad \sum_{j=1}^{\infty} |f_j(x)| < \infty \end{cases}$$

Finally to prove (iii), we observe that for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{j=N}^{\infty} \int |f_j|\, dx < \epsilon.$$

Since

$$f - f_1 - f_2 - ... - f_N \sim f_{N+1} + f_{N+2} + ...$$

it follows that

$$\int \left| f - \sum_{j=1}^{N} f_j \right| \, dx \leq \sum_{j=N+1}^{\infty} \int |f_j| < \epsilon.$$

Thus $||f - \sum_{j=1}^{N} f_j|| \to 0$ as $N \to \infty$. □

To finish the proof of Theorem 4.2 we need a lemma about Banach spaces

Lemma 4.2. *Let X be a normed vector space. The following are equivalent*
i) X is complete (that is, every Cauchy sequence converges)
ii) Every absolutely convergent series converges. i.e.

$$\textit{if} \quad \sum_{n=1}^{\infty} ||x_n|| < \infty \Longrightarrow \sum_{n=1}^{\infty} x_n = x \in X.$$

Proof. (i) implies (ii). Suppose X is complete and let $\sum_{n=1}^{\infty} ||x_n|| < \infty$. Let $\epsilon > 0$ Take $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} ||x_n|| < \epsilon$. Then the sequence of partial sums $S_k = x_1 + ... + x_k$ satisfies for $M, L > N$

$$||S_M - S_L|| \leq \sum_{n=N}^{\infty} ||x_n|| < \epsilon.$$

Thus S_k is a Cauchy sequence and hence converges.

To prove that (ii) implies (i) Let x_n, $n = 1, 2, ...$ be a Cauchy sequence in X and we want to prove that (ii) imples that it converges. To prove that it is only necessary to prove that it has a convergent subsequence. To construct this subsequence, notice that, as x_n is Cauchy, for every $k \in \mathbb{N}$ there exists $N(k) \in \mathbb{N}$, which can be taken to be increasing in k, such that

$$||x_m - x_n|| < 2^{-k} \ \ \forall \ \ m, n \geq p_k.$$

Then the series

$$\sum_{k=1}^{\infty} ||x_{N(k+1)} - x_{N(k)}|| \leq \sum_{k=1}^{\infty} 2^{-k} < \infty$$

and (ii) implies that $\sum_{k=1}^{\infty} x_{N(k+1)} - x_{N(k)} = \lim_{k\to\infty} x_{N(k)} = x$. This ends the proof of the lemma. □

To prove Theorem 4.2 notice that Lemma 4.1 shows that property (ii) of Lemma 4.2 holds for $L^1(\mathbb{R})$.

5. Convergence Theorems

This section expresses the power of the Lebesgue integration and the completeness of $L^1(\mathbb{R})$.

We say that a certain property of a set of functions holds almost everywhere, and denote a.e., when it holds for all $x \in \mathbb{R}$, except possibly on a null set. For example we say that $f = g$ a.e. if $f(x) = g(x)$ for all $x \in \mathbb{R}$ with the possible exception of x in a null set. Similarly a sequence $f_n \to f$ a.e. if $f_n(x) \to f(x)$ for all $x \in \mathbb{R}$ with the possible exception of x in a null set. We have the following property of convergence in $L^1(\mathbb{R})$.

Theorem 5.1. *Suppose $f_n \in L^1(\mathbb{R})$, $n = 1, 2, ...$ and $f_n \to f$ in $L^1(\mathbb{R})$. Then there exists a subsequence f_{nk} of f_n such that $f_{nk} \to f$ a.e.*

Proof. Since $\int |f_n - f| \, dx \to 0$ as $n \to \infty$, then for every k there exists an integer $N(k)$ such that

$$\int |f_{N(k)} - f| \, dx \leq 2^{-k}$$

and $N(k)$ can be taken to be increasing in k. Let

$$g_1 = f_{N(1)}, \quad g_k = f_{N(k+1)} - f_{N(k)}.$$

Then

$$\int |g_k| \, dx \leq \int |f_{N(k+1)} - f| \, dx + \int |f_{N(k)} - f| \, dx \leq \frac{3}{2} 2^{-k}$$

and therefore

$$\sum_{k=1}^{\infty} \int |g_k| \, dx < \infty$$

Hence by Lemma 4.1 there exists $g \in L^1(\mathbb{R})$ such that

$$||g - \sum_{k=1}^{M} g_k|| \to 0 \quad \text{as} \quad M \to \infty \quad \text{and} \quad g = \sum_{k=1}^{\infty} g_k \quad \text{a.e.}$$

Thus

$$f_{N(k)} \to g \quad \text{in} \quad L^1(\mathbb{R}) \ \text{ and a.e.}$$

But since $\int |f_{N(k)} - f| \, dx \le \frac{3}{2} 2^{-k}$, $\int |f - g| \, dx = 0$ and hence $f = g$ a.e. □

The converse of this theorem is not true. For example, let

$$f_n(x) = \begin{cases} \frac{1}{\sqrt{n}} & \text{if} \quad -n \le x \le n \\ 0 & \text{if} \quad |x| > n \end{cases}$$

Then $f_n(x) \to 0$ for all $x \in \mathbb{R}$, but $\int f_n \, dx \to \infty$.

Theorem 5.2. *(The monotone convergence Theorem) Let $f_n \in L^1(\mathbb{R})$, $n = 1, 2, ...$ satisfying*
(H1) f_n is monotone a.e., that is either $f_n(x) \le f_{n+1}(x)$, a.e., or $f_n(x) \ge f_{n+1}(x)$, a.e.
(H2) There exists $M > 0$ such that $|\int f_n \, dx| < M$, for all n.
Then there exists $f \in L^1(\mathbb{R})$ such that $f_n \to f$ in $L^1(\mathbb{R})$ and $f_n \to f$ a.e.

Proof. We may assume that $f_1 \le f_2 \le ...$, otherwise just consider the sequence $-f_n$. By subracting f_1 from each element of the sequence we may assume that

$$0 \le f_1 \le f_2 \le ...$$

Let

$$g_1 = f_1, \quad g_n = f_{n+1} - f_n, \quad n = 2, 3...$$

Then $g_n \ge 0$ and

$$\sum_{n=1}^{k} \int g_n \, dx = \int g_k \, dx \le M.$$

Hence by Lemma 4.1 there exists $g \in L^1(\mathbb{R})$ such that $g_n \to g$ in $L^1(\mathbb{R})$ and a.e. □

Theorem 5.3. *(Fatou's Lemma) Let $f_n \in L^1(\mathbb{R})$ $n = 1, 2, ...$ $f_n \ge 0$ is such that*

$$\int f_n \, dx < M, \quad n = 1, 2, ...$$

If $f_n \to f$ a.e, then $f \in L^1(\mathbb{R})$ and $\int f \, dx \le M$.

Proof. For $m, n \in \mathbb{N}$, let

$$g_{n,k} = \min\{f_n, ..., f_{n+K}\}.$$

Then $g_{n,k} \in L^1(\mathbb{R})$ (why?) and for n fixed

$$g_{n,k} \ge g_{n,k+1}.$$

By the monotone convergence theorem there exists $G_n \in L^1(\mathbb{R})$ such that

$$\lim_{k \to \infty} g_{n,k} = G_n \quad \text{in} \quad L^1(\mathbb{R}) \quad \text{and } a.e.$$

But

$$G_n(x) = \inf\{f_j(x) : \quad j \geq n\} \tag{5.1}$$

and hence

$$G_n \leq G_{n+1}$$

satisfy $\int G_n \, dx < \int f_n \, dx \leq M$. Again by the monotone convergence theorem, there exists $G \in L^1(\mathbb{R})$ such that

$$G_n \to G \quad \text{in} \quad L^1(\mathbb{R}) \quad \text{and } a.e.$$

In view of (5.1), if $f_n(x) \to f(x)$ for some x, $G(x) = f(x)$. Thus $G(x) = f(x)$ a.e. and $\int f \, dx \leq M$. □

In general it is not true that $\lim \int f_n \, dx = \int f \, dx$, as can be seen by the example

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if} \quad -n \leq x \leq n \\ \\ 0 & \text{if} \quad |x| > n \end{cases}$$

Theorem 5.4. *(The dominated convergence theorem) Let $f_n \in L^1(\mathbb{R})$, $n = 1, 2...$ satisfy:*
(H1) $f_n \to f$ a.e.
(H2) There exists $h \in L^1(\mathbb{R})$ such that $|f_n(x)| \leq h(x)$ a.e. $n = 1, 2...$
Then $f \in L^1(\mathbb{R})$ and $f_n \to f$ in $L^1(\mathbb{R})$.

Proof. For $m, n \in \mathbb{N}$ let

$$g_{m,n} = \max\{|f_m|, ..., |f_{m+n}|\} \leq h \text{ a.e.}$$

For m fixed

$$g_{m,n}(x) \leq g_{m,n+1}(x) \quad \text{and} \quad \left|\int g_{m,n} \, dx\right| < \int h \, dx < \infty.$$

By the monotone convergence theorem there exists $g_m \in L^1(\mathbb{R})$ such that

$$\lim_{n\to\infty} g_{m,n} = g_m \quad \text{a.e. and in} \quad L^1(\mathbb{R}).$$

Since $g_m(x) = \sup\{|f_j(x)|, \quad j \geq m\} \leq h(x)$, a.e., it follows that $g_m(x) \geq g_{m+1}(x) \geq 0$ a.e. and $|\int g_m \, dx| \leq \int h \, dx$. Thus by the monotone convergence theorem there exists $g \in L^1(\mathbb{R})$ such that $g_m(x) \to g(x)$ in $L^1(\mathbb{R})$ and a.e.

If $f_n \to 0$ a.e, then $g_m \to 0$ a.e Since

$$\int |f_m| \, dx \leq \int g_m \, dx \to 0$$

$f_n \to 0$ in $L^1(\mathbb{R})$.

For $r, s \in \mathbb{N}$,

$$h_{r,s} = f_r - f_s \to 0 \quad \text{a.e as} \quad r, s \to \infty$$
$$|h_{r,s} \leq 2|h| \ \ \forall \ r, s.$$

Hence by the case above, $|h_{r,s}| \to 0$ in $L^1(\mathbb{R})$. This shows that f_n is a Cauchy sequence in $L^1(\mathbb{R})$ and hence $f_n \to F$ in $L^1(\mathbb{R})$. But by Theorem 5.1 there exists a subsequence of f_n that converges to F a.e. and therfore $F = f$ a.e. Thus $f \in L^1(\mathbb{R})$ and $f_n \to f$ in $L^1(\mathbb{R})$. □

6. The space $L^1_{\text{loc}}(\mathbb{R})$

Theorem 6.1. *Let $f \in L^1(\mathbb{R})$ and $[a,b] \subset \mathbb{R}$ then $\chi_{[a,b]} f \in L^1(\mathbb{R})$*

Proof. Let f_j, $j = 1, 2...$ be step functions such that $f \sim f_1 + f_2 + ...$ Let $g_j = \chi_{[a,b]} f_j$. Then g_j is a step function and $f\chi_{[a,b]} \sim g_1 + g_2 + ...$ □

The converse of this result is certainly not true.

Definition 6.1. *A function f is locally integrable $f \in L^1_{\text{loc}}(\mathbb{R})$ if for every interval $[a,b]$*

$$\int_a^b f\, dx = \int f\chi_{[a,b]}\, dx \quad \textit{exists.}$$

In this case we say that $f \in L^1_{\text{loc}}(\mathbb{R})$.

Theorem 6.2. *Let $f, g \in L^1_{\text{loc}}(\mathbb{R})$ If g is bounded on every interval $[a,b] \subset \mathbb{R}$ then $fg \in L^1_{\text{loc}}(\mathbb{R})$.*

Proof. Let $F = \chi_{[a,b]} f$ and $G = \chi_{[a,b]} g$. Then there exist step functions f_n, $n = 1, 2, ...$ such that $F \sim f_1 + f_2 + ...$ We know from Theorem 6.1 that $Gf_j \in L^1(\mathbb{R})$. Moreover if $|G| \leq M$

$$\int |Gf_j|\, dx \leq M \int |f_j|\, dx$$

so we get that $\sum_{j=1}^{\infty} \int |Gf_j|\, dx < \infty$. Morover $\sum_{j=1}^{\infty} |G(x)f_j(x)| < \infty$ if and only if either $\sum_{j=1}^{\infty} |f_j(x)| < \infty$ or $G(x) = 0$. This shows that

$$G(x)F(x) = \sum_{j=1}^{\infty} G(x)f_j(x) \ \ \forall\ x \quad \text{such that} \quad \sum_{j=1}^{\infty} |G(x)f_j(x)| < \infty.$$

□

Theorem 6.3. *If $f \in L^1_{\text{loc}}(\mathbb{R})$ and $|f| \leq g$ for some $g \in L^1(\mathbb{R})$ then $f \in L^1(\mathbb{R})$.*

Proof. Let $f_n = f\chi_{[-n,n]}$, $n = 1, 2...$ Then $f_n \in L^1(\mathbb{R})$ and $|f_n| \leq g$ thus by the dominated convergence theorem $f = \lim_{n\to\infty} f_n \in L^1(\mathbb{R})$. □

7. The Lebesgue Measure

Definition 7.1. *A set $S \subset \mathbb{R}$ is measurable if $\chi_S \in L^1_{\text{loc}}(\mathbb{R})$. If S is measurable and $\chi_S \in L^1(\mathbb{R})$ we define the measure of S by $m(S) = \int \chi_S\, dx$. Otherwise we say that $m(S) = \infty$.*

Theorem 7.1. *Let $\mathcal{M}(\mathbb{R})$ denote the collection of all measurable sets in $\mathbb{R}$. Then*
1)$S_1, S_2 \in \mathcal{M}(\mathbb{R})$ then $S_2 \setminus S_1 \in \mathcal{M}(\mathbb{R})$.
2) $S_j \in \mathcal{M}(\mathbb{R})$, $j = 1, 2, ...N$, then $S_1 \cup S_2 \cup ... \cup S_N \in \mathcal{M}(\mathbb{R})$.
3) If $S_j \in \mathcal{M}(\mathbb{R})$, $j = 1, 2, ...$ and $S_i \cap S_j = \emptyset$ if $i \neq j$ then

$$m\left(\bigcup_{j=1}^{\infty} S_j\right) = \sum_{j=1}^{\infty} m(S_j).$$

4) If $S_j \in \mathcal{M}(\mathbb{R})$, $j = 1, 2, ...$

$$m\left(\bigcup_{j=1}^{\infty} S_j\right) \leq \sum_{j=1}^{\infty} m(S_j).$$

5) If $A, B \in \mathcal{M}(\mathbb{R})$ and $A \subset B$ then $m(B \setminus A) = m(B) - m(A)$.
6) If $S_j \in \mathcal{M}(\mathbb{R})$, $j = 1, 2, \ldots$ and $S_1 \subset S_2 \subset \ldots$ then

$$m\left(\bigcup_{j=1}^{\infty} S_j\right) = \lim_{n\to\infty} m(S_j).$$

7) If $S_j \in \mathcal{M}(\mathbb{R})$, $j = 1, 2, \ldots$ and $S_1 \supset S_2 \supset \ldots$ then

$$m\left(\bigcap_{j=1}^{\infty} S_j\right) = \lim_{n\to\infty} m(S_j).$$

More tomorrow....

Department of Mathematics, Purdue University, 150 North University Street, West Lafayette IN 47907, USA

E-mail address: sabarre@math.purdue.edu