LOWER BOUNDS FOR THE NUMBER OF RESONANCES IN EVEN DIMENSIONAL POTENTIAL SCATTERING

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1. Introduction and Statement of Results

In this article we establish a lower bound for the number of resonances for the perturbation of the Laplacian in \( \mathbb{R}^n \), \( n \geq 4 \) even, by any non-zero real valued potential \( V \in C_0^\infty(\mathbb{R}^n, \mathbb{R}) \). The resonances are defined as the poles of the meromorphic continuation of the resolvent or the scattering matrix.

It was shown in [13] that there exists at least one resonance. The results proved here in particular show that there exist infinitely many. This seems to be the first lower bound for the number of resonances in even dimensions that holds for any non-zero \( V \in C_0^\infty(\mathbb{R}^n, \mathbb{R}) \). In the odd dimensional case lower bounds have been obtained by Christiansen [3]. In fact the proof of Theorem 1.1 below is inspired by [3]. Here we combine the methods of [13] and the fundamental ideas of [22] to prove the main result of this article.

**Theorem 1.1.** Let \( V \in C_0^\infty(\mathbb{R}^n; \mathbb{R}) \), \( n \geq 4 \) even, and \( P = -\Delta + V \). If \( V \neq 0 \), then the poles \( \{ \lambda_j \} \), \( \lambda_j = |\lambda_j|e^{i\arg \lambda_j}, |\lambda_j| > 0 \), with multiplicity \( M(\lambda_j) \), of the meromorphic continuation of \((P - \lambda^2)^{-1} : L^2_{\text{comp}}(\mathbb{R}^n) \rightarrow H^2_{\text{loc}}(\mathbb{R}^n)\), \( \exists \lambda > 0, \lambda^2 \not\in \sigma(P) \), to the logarithmic plane \( \Lambda \) satisfy

\[
\sum_{\text{poles}} \frac{M(\lambda_j)}{\log|\lambda_j| + i \arg \lambda_j} = \infty.
\]

A direct consequence of this result is

**Corollary 1.1.** Let \( V \neq 0 \) satisfy the hypotheses of Theorem 1.1. Let \( N(r) \) denote the number of poles, counted with multiplicity, satisfying \( \frac{1}{p} < |\lambda| < r, |\arg(\lambda)| < \log r \). Then

\[
\limsup_{r \to \infty} \frac{N(r)}{(\log r)(\log \log r)^{-p}} = \infty, \quad \forall p > 1.
\]

As in [3], [14], [15] and [13], the method we use is not constructive and is still far from the optimal upper bound obtained by Vodev [20], [21], which states that the counting function

\[
N(r,a) = \# \{ \lambda_j : \text{a pole counted with multiplicity,} \ 0 < |\lambda_j| < r, \ |\arg \lambda_j| < a \}
\]

satisfies \( N(r,a) \leq Ca \ (r^n + (\log a)^n) \), \( n \geq 2 \). Intissar [7] had previously established that \( N(r) \leq Cr^{n+1} \), for \( n \geq 4 \). It is clear that \( N(r,\log r) \geq N(r) \). We remark that the bound (1.2) does not distinguish between poles occurring on different sheets of \( \Lambda \). In particular it does not guarantee the existence of infinitely many poles on every sheet.

The sharp upper bound for the odd dimensional scattering by a potential was obtained by Zworski [23]. For a survey on pole counting we refer the reader to [24], see also [11], [22], [14], [15] and [13].

2. Proof of Theorem 1.1

In the even dimensional case the scattering matrix \( S_V(\lambda) \) is an operator valued meromorphic function defined on the logarithmic plane \( \Lambda \). We observe that according to equation (1.3) and Theorem 1 of [18],
$S_V(\lambda)$ satisfies

$$S_V(\lambda)S_V(\overline{\lambda})^* = I, \quad S_V(\lambda) = 2I - S_V(e^{i\pi \lambda}), \quad \text{where for } \lambda = |\lambda|e^{i\text{arg}(\lambda)}, \overline{\lambda} = |\lambda|e^{-i\text{arg}(\lambda)}.$$  

Thus if $\lambda$ is a pole of $S_V$, then $\overline{\lambda}$ is a zero and $\text{exp}(-i\pi \lambda) = |\lambda|\text{exp}(-i(\text{arg}(\lambda) - \pi))$ is also a pole. Observe that, unless $\text{arg}(\lambda) = \pi/2$, $\lambda \neq \overline{\lambda}\text{exp}(-i\pi)$. This is the analogue of the fact that in odd dimensions the poles are symmetric with respect to the imaginary axis. Poles of $S_V(\lambda)$ that satisfy $\text{arg}(\lambda) = \pi/2$ are square roots of negative eigenvalues.

We know from [9] and [18], see also equation (3.7) of [22], that, with possibly finitely many exceptions, in case there are eigenvalues or zero resonances, the poles of the meromorphic continuation of the resolvent coincide with multiplicity with those of the determinant of the scattering matrix. Since $S_V(\lambda)$ is a meromorphic function on $\Lambda$, it follows that by setting $\lambda = e^z, z \in \mathbb{C}$, the function $S(z) = \det S_V(e^z)$ is a meromorphic function on the complex plane $\mathbb{C}$. It follows from (2.1) that if $z_j$ is a pole of $S(z)$, then $\overline{z_j}$ is a pole of $S(\overline{z})$. Since there are no poles on the ray $\text{arg } \lambda = 0$, there are no poles $z_j$ with $\Im z_j = 0$. As in [13] we use some of the results of [7] and the methods of [22] to prove

**Proposition 2.1.** Let $S_V(\lambda), \lambda \in \Lambda$, be the scattering matrix for $-\Delta + V, V \in C_0^\infty(\mathbb{R}^n)$, real valued and $n \geq 4$ even. Let $\{z_j\}$ denote the poles of $S(z) = \det S_V(e^z)$ with multiplicity $M(z_j)$. If there exists $m \in \mathbb{R}, m \geq 1,$ such that

$$\sum_j \frac{M(z_j)}{|z_j|^m} < \infty,$$

then

$$S(z) = e^{g(z)}\frac{P(z, m)}{Q(z, m)} \quad \text{if } m > 1, \quad S(z) = e^{g(z)}\frac{P(z, 0)}{Q(z, 0)} \quad \text{if } m = 1,$$

where

$$P(z, m) = \prod \left(1 - \frac{z}{z_j}\right)^{M(z_j)} \exp \left(M(z_j) \sum_{k=1}^{|m|} \frac{1}{k} \left(\frac{z}{z_j}\right)^k\right),$$

$$Q(z, m) = \prod \left(1 - \frac{z}{z_j}\right)^{M(z_j)} \exp \left(M(z_j) \sum_{k=1}^{|m|} \frac{1}{k} \left(\frac{z}{z_j}\right)^k\right),$$

$$P(z, 0) = \prod \left(1 - \frac{z}{z_j}\right)^{M(z_j)}, \quad Q(z, 0) = \prod \left(1 - \frac{z}{z_j}\right)^{M(z_j)},$$

$[m]$ is the greatest integer strictly less than $m$, and $g(z)$ is an entire function satisfying

$$|g(z)| \leq C \exp(C|z|), \quad C > 0.$$  

We postpone the proof of Proposition 2.1 until the next section. We will use it to prove the following Proposition, which easily gives Theorem 1.1.

**Proposition 2.2.** Let $S(z)$ and $\{z_j\}$ be as in Proposition 2.1. Then

$$\sum_j \frac{M(z_j)}{|z_j|} = \infty.$$  

**Proof.** As in [3] and [13] the proof is by contradiction. We will prove that if (2.5) does not hold, the expansion of the scattering phase $\sigma(\lambda)$ as $\lambda \to 0^+$ and $\lambda \to +\infty$ can not be satisfied unless $V = 0$. In which case, of course, there are no poles.
We know from [12], see also [5], that $\sigma(\lambda) = \frac{1}{2\pi i} \log \det S_V(\lambda)$, $\lambda > 0$, satisfies
\begin{equation}
\sigma'(\lambda) = \sum_{j=1}^{\frac{d}{2} - 1} \alpha_j(V)\lambda^{n-2j} + O(\lambda^{-\infty}), \quad \text{as} \quad \lambda \to +\infty.
\end{equation}

We remark that the error $O(\lambda^{-\infty})$ in (2.6) is due to the fact that, in even dimensions, the trace of regularized the wave group has an expansion near $t = 0$, in even powers of $t$, with only finitely many singular terms, see for example Theorem 17.5.5 of [6].

If $\sum \frac{M(z)}{|z|^2} < \infty$, it follows that (2.2) holds with $m = 1$, and we deduce from (2.3) that $\theta(z) = \frac{1}{2\pi i} \log S(z)$ satisfies
\begin{equation}
2\pi i \theta'(z) = g'(z) + \frac{P'(z,0)}{P(z,0)} - \frac{Q'(z,0)}{Q(z,0)} = g'(z) + \sum_j M(z_j) \left( \frac{1}{z_j - z} - \frac{1}{\overline{z}_j - \overline{z}} \right)
\end{equation}

On the other hand, since for $\Im z = 0$, $\theta(z) = \sigma(\pm z)$, it follows from (2.6) that
\begin{equation}
\theta'(z) = e^z \sigma'(\pm z) = \sum_{j=1}^{\frac{d}{2} - 1} \alpha_j(V) e^{(n-2j)z} + O((\pm z)^{-\infty}), \quad \text{as} \quad \Im z = 0, \quad \Re z \to +\infty.
\end{equation}

Let
\begin{equation}
F(z) = P(z,0)Q(z,0) \left( \theta'(z) - \sum_{j=1}^{d-1} \alpha_j(V)e^{(n-2j)z} \right), \quad z \in \mathbb{C}
\end{equation}

Under the assumption that $\sum \frac{M(z)}{|z|^2} < \infty$ we deduce that (see for example Page 286 of [19])
\begin{equation}
|P(z,0)| \leq C_\epsilon \exp(\epsilon|z|), \quad |Q(z,0)| \leq C_\epsilon \exp(\epsilon|z|), \quad \forall \quad \epsilon > 0.
\end{equation}

It follows from Cauchy’s formula (see Theorem 8.5.1 of [19],) that
\begin{equation}
|P'(z,0)| \leq C_\epsilon \exp(2\epsilon|z|), \quad |Q'(z,0)| \leq C_\epsilon \exp(2\epsilon|z|), \quad \forall \quad \epsilon > 0.
\end{equation}

Hence (2.4), (2.7), (2.8), (2.9) and (2.10) give that $F(z)$ is an entire function which satisfies
\begin{equation}
|F(z)| \leq C e^{C|z|}, \quad z \in \mathbb{C},
\end{equation}
\begin{equation}
|F(z)| \leq C e^{-k|z|}, \quad \forall \quad k \in \mathbb{N}, \Re z = 0, \quad \Re z \to +\infty.
\end{equation}

For the constant $C$ in (2.11), let $F_1(z) = F(z)\exp((C+1)z)$. Then
\begin{equation}
|F_1(z)| \leq C e^{(2C+1)|z|}, \quad z \in \mathbb{C},
\end{equation}
\begin{equation}
|F_1(z)| \leq C e^{-|z|}, \quad \Im z = 0, \quad \Re z \to \pm \infty.
\end{equation}

It is an application of the Phragmén-Lindelöf principle, which is due to Carlson, see section 3 of [13], see also Theorem 5.8 of [19], that an entire function which satisfies these two properties must be identically zero. Thus
\begin{equation}
\theta'(z) = \sum_{j=1}^{\frac{d}{2} - 1} \alpha_j(V)e^{(n-2j)z}.
\end{equation}

In particular this gives that $\theta'(z)$ is entire and, since $g(z)$ is entire and $\Im z \neq 0$, (2.7) implies that there can be no poles of $S_V(\lambda)$ in $\Lambda$. It follows from (2.12) and (2.8) that
\begin{equation}
\sigma'(\lambda) = \sum_{j=1}^{d-1} \alpha_j(V)\lambda^{n-2j}, \quad \lambda > 0.
\end{equation}
Since $0 \not\in \Lambda$ it has to be analyzed separately. So far we have proved that if (2.5) does not hold, $S_V(\lambda)$ has no poles in $\Lambda$ and $\sigma'(\lambda)$ is given by (2.13). We claim that these two facts imply that $\lambda = 0$ is neither an eigenvalue nor a resonance.

Indeed, since there are no poles in $\Lambda$, in particular there are no negative eigenvalues, and since $V \in C_0^\infty(\mathbb{R}^n)$, it follows that $0$ cannot be an eigenvalue. Otherwise $-\Delta + V$ would have a ground state with zero energy (see Theorem XIII. 44 of [16]) which does not exist for smooth compactly supported potentials, see for example [17], [2]. While the results in [17] and [2] are stated for $n = 3$, it is easy to see that the proof in [2] works in any dimension, at least for compactly supported potentials. Thus, since $0$ is not a resonance in dimensions $n \geq 5$, the claim holds in even dimensions $n \geq 6$.

The case $n = 4$ has to be analyzed more carefully. From the discussion above we know that $0$ is not an eigenvalue. We will show that if $0$ is a resonance the expansion (2.13) can not be satisfied. We recall from equation (4.5) of [22], see also equation (3.13) of [7], that for $n = 4$,

$$S_V(\lambda) = I + A_V(\lambda)$$

where

$$A_V(\lambda)(\omega, \theta) = C \lambda^2 \int_{\mathbb{R}^n} e^{-i \lambda \langle x, \omega \rangle} V(x) \left[ (I - R_0(\lambda)V)^{-1} (e^{i \lambda \langle \cdot, \theta \rangle}) \right](x)dx.$$  

Since $\sigma(\lambda) = \frac{1}{2\pi i} \log \det S_V(\lambda)$, $\lambda > 0$, it follows from (2.1) that

$$\sigma'(\lambda) = \frac{1}{2\pi i} \text{Trace } S_V(\lambda)^{*} S'(\lambda) = \frac{1}{2\pi i} \text{Trace } ((I - A_V(-\lambda))A'_V(\lambda)).$$

Now we refer to the expansion of $(I - R_0(\lambda)V)^{-1}$ as $\lambda \to 0^+$ established in [8]. In the terminology of [8], $0$ is an exceptional point of the first kind if it is a resonance but not an eigenvalue. In this case there exists a unique resonance function $\psi$ normalized so that $\langle V\psi, 1 \rangle = \int V(x)\psi(x)dx = 4\pi$. Lemma 4.3 of [8] gives that

$$(I - R_0(\lambda)V)^{-1} = \lambda^{-2}(a - 2\log \lambda)^{-1}\langle \cdot, V\psi \rangle\psi + B + O\left(\frac{1}{\log \lambda}\right), \quad a \neq 0, \quad \lambda \to 0^+$$

Substituting (2.16) and (2.14) in (2.15) we find that

$$\sigma'(\lambda) = \frac{C}{\lambda(a - 2\log \lambda)^2} + O\left(\frac{1}{\log \lambda}\right), \quad C \neq 0.$$

Thus if $0$ is a resonance (2.13) can not hold.

So we conclude that if (2.5) does not hold $S_V(\lambda)$ has no poles in $\Lambda$, $0$ is neither an eigenvalue nor a resonance and (2.13) holds. In this case equation (3.3) of [25] states that

$$\sigma'(\lambda) = \lambda^{n-3} f(\lambda, \lambda^{n-2}\log(\lambda)), \quad \lambda > 0, \quad f \in C^\infty.$$

Thus $\sigma'(\lambda)$ vanishes to order $n - 3$ as $\lambda \to 0^+$, and hence it follows from (2.13) that

$$\sigma'(\lambda) = \alpha_1(V)\lambda^{n-3}, \quad \lambda > 0.$$  

Again using the fact that $-\Delta + V$ has no negative eigenvalues and 0 is neither an eigenvalue nor a resonance, we deduce from (2.17) and equation (3.17) and equation (3.3) of [5] that the regularized heat trace satisfies

$$H(t) = \text{Trace } \left( e^{-t(-\Delta + V)} - e^{-t\Delta} \right) = \int_0^\infty e^{-t\lambda} \frac{d}{d\lambda} \sigma(\sqrt{\lambda})d\lambda =$$

$$\int_0^\infty \frac{1}{2} \alpha_1(V)\lambda^{n-2} e^{-t\lambda}d\lambda = \frac{1}{2^n} \left(\frac{n}{2} - 1\right) \alpha_1(V)t^{1-\frac{n}{2}}.$$
But it is well known that $H(t)$ has an asymptotic expansion as $t \to 0^+$ given by

$$H(t) \sim \sum_{j=1}^{\infty} C_j(V) t^{j-\frac{d}{2}}, \quad t \to 0^+.$$  

In particular (2.18) gives that $C_2(V) = 0$. But, see for example [1], $C_2(V) = \frac{1}{2} \int_{\mathbb{R}^n} V^2(x)dx = 0$. Hence $V$ must be identically zero. This concludes the proof of the Proposition. \(\square\)

To prove Theorem 1.1 just observe that if $\lambda_j$ is a pole of $\det S_V(\lambda)$ with multiplicity $M(\lambda_j)$ and $\lambda_j = e^{z_j}$, then $z_j = \log |\lambda_j| + i \arg(\lambda_j)$ is a pole of $S(z)$ with the same multiplicity. Thus (1.1) follows from (2.5).

To prove Corollary 1.1, observe that if (1.2) does not hold, then there exist $C > 0$ and $p > 1$ such that

$$C \log r (\log \log r)^{-p} \geq N(r) \geq \# \{z_j = \log |\lambda_j| + i \arg(\lambda_j); |z_j| < \log r\}.$$  

Therefore

$$\# \{z_j = \log |\lambda_j| + i \arg(\lambda_j); |z_j| < r\} \leq C r (\log r)^{-p}.$$  

But this implies that $\sum_{|\lambda_j| > r} \frac{M(\lambda_j)}{|\log |\lambda_j| + i \arg(\lambda_j)|} < \infty$, which contradicts (1.1).

3. Upper Bounds on the Determinant of the Scattering Matrix

Now we prove Proposition 2.1.

**Proof.** Since $\{\overline{\pi}\}$ and $\{\pi\}$ are respectively the zeros and poles of $S(z)$, then $S(z)$ can be expressed as in (2.3). All we need to prove is the bound (2.4).

We follow the proof of Proposition 2.1 of [13], which in turn follows that of Proposition 6 of [22]. Let $R_0(\lambda)$ denote the holomorphic continuation of $(-\Delta - \lambda^2)^{-1}$ to $\Lambda$. Again we recall equations (4.5) of [22], and equation (3.13) of [7]. They give that

$$S_V(\lambda) = I + A_V(\lambda)$$  

where

$$A_V(\lambda) = E^\rho(-\lambda)V(I - H_V(\lambda))^{-1}E^\rho(\lambda),$$  

$$H_V(\lambda) = \rho R_0(\lambda)V, \quad \rho \in C_0^\infty, \quad \rho V = V$$  

and $E^\rho(\lambda)$ has Schwartz kernel given by

$$C_n \lambda^{\frac{n-2}{2}} e^{i\lambda(x, \omega)} \rho(x).$$  

Let $\mu_j(A(\lambda))$ denote the characteristic values of $A_V(\lambda)$, then

$$\det S_V(\lambda) \leq \prod_{j=1}^{\infty} (1 + \mu_j(A_V(\lambda))).$$  

To estimate $\mu_j(A(\lambda))$ we use that

$$\mu_j(A(\lambda)) \leq \mu_j(E^\rho(-\lambda)) ||V||. \|(I - H_V(\lambda))^{-1}|| \||E^\rho(\lambda)||.$$  

Setting $\lambda = e^z$, the argument used in [22], see also the proof of Corollary 1.1 of [7], and Proposition 2 of [23], shows that

$$\mu_j(E^\rho(e^z)) \leq C \exp(\exp C|z| - \frac{1}{4} \pi^2/C), \quad C > 0.$$  

It is easy to see that

$$||E^\rho(e^z)|| \leq C \exp(\exp C|z|).$$
To estimate \( \| (I - H_V(e^z))^{-1} \| \) we use, as in [22], Theorem V.5.1 of [4], i.e.,
\[
(3.5) \quad \| (I - H_V(e^z))^{-1} \| \leq \left| \det \left( I + (H_V(e^z))^{\frac{1}{2} + 1} \right) \right|^{-1} \cdot \left| \det (I + [H_V(e^z)]^{\frac{1}{2} + 1}) \right|^{-1}.
\]
From Proposition 2.1 of [7] we have that
\[
(3.6) \quad \left| \det \left( I + (H_V(e^z))^{\frac{1}{2} + 1} \right) \right| \leq \det (I + [H_V(e^z)]^{\frac{1}{2} + 1}) \leq \exp(\exp(n + 1)|z|).
\]
Cartan’s estimate, see for example Theorem I.11 of [10], states that if \( f(z) \) is a holomorphic function in the disk \( |z| \leq 2R, \) with \( f(0) = 1, \) and \( \eta \in (0, \frac{3}{2R}) \), then outside a family of disks the sum of whose radii is not greater than \( 4\eta R, \) we have for these values of \( \eta \)
\[
\log |f(z)| > -\left( 2 + \log \frac{3e}{2\eta} \right) \log M(2eR),
\]
\[
M(s) = \sup \{|f(z)| : |z| \leq s\}.
\]
Then for \( \eta = \frac{1}{2R} \) there exists \( \rho \in (\frac{R}{4}, R) \) such that the boundary of the disk of center zero and radius \( \rho \) does not intersect any of the excluded disks. Otherwise the sum of their radii would have to be greater than or equal to \( R/4. \) Since \( R < 2\rho \) we obtain, for \( m(\rho) = \inf \{|f(z)| : |z| = \rho\}, \)
\[
(3.7) \quad \log m(\rho) > -2 + \log(3e) \log M(4\rho).
\]
Suppose that \( \left| \det \left( I + (H_V(e^z))^{\frac{1}{2} + 1} \right) \right| = az^p f(z), \) where \( f(z) \) is entire and \( f(0) = 1. \) Then it follows from (3.6) and (3.7) that for every \( R > 0, \) there exists \( \rho \in (R/2, R) \) such that
\[
(3.8) \quad \left| \det \left( I + (H_V(e^z))^{\frac{1}{2} + 1} \right) \right| \geq K(p) \exp(-C \exp(4e(n + 1)|z|)), \quad |z| = \rho.
\]
Hence from (3.5), (3.6) and (3.8), we have, for these values of \( \rho, \)
\[
(3.9) \quad \| (I - H_V(e^z))^{-1} \| \leq K(p) \exp(2C \exp(4(n + 1)e|z|)), \quad |z| = \rho.
\]
We deduce from (3.2), (3.3), (3.4) and (3.9) that there exists \( C > 0 \) such that
\[
(3.10) \quad \mu_j (A_V(e^z)) \leq C \exp(\exp(C|z| - \frac{1}{C} j^{1/m})), \quad |z| = \rho.
\]
Then, as in [22], see also [7] or [14], it follows from (3.1) and (3.10) that
\[
|S(z)| \leq C \exp(\exp(C|z|)), \quad |z| = \rho.
\]
Since \( \sum \frac{M(z)}{\log(\rho_m)} < \infty \) it follows (see Theorem 8.25 of [19]) that
\[
(3.11) \quad |Q(z,m)| \leq C \exp(C|z|^{m+\epsilon}), \quad |P(z,m)| \leq C \exp(C|z|^{m+\epsilon}), \quad \forall \epsilon > 0.
\]
Thus \( |Q(z,m)S(z)| \leq C \exp(\exp(C|z|)), \quad |z| = \rho. \) But since \( Q(z,m)S(z) \) is an entire function, it follows from the maximum principle that \( |Q(z,m)S(z)| = \exp(g(z))P(z,m)| \leq C \exp(\exp(C|z|)), \) for \( z \in \mathbb{C}. \) Applying (3.7) to \( f(z) = P(z,m) \) we find that for every \( R > 0, \) there exists \( \rho' \in (\frac{R}{2}, R) \) for which
\[
(3.12) \quad \exp(g(z)) \leq C \exp(\exp(C|z|)), \quad |z| = \rho'.
\]
Since \( g \) is entire, the maximum principle guarantees that there exists \( C > 0 \) such that (3.12) holds for all \( z \in \mathbb{C}. \) Then (2.4) follows from Borel-Carathéodory theorem (see Theorem 5.5 of [19]).

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