DETERMINING ASYMPTOTICS OF MAGNETIC FIELDS FROM FIXED ENERGY SCATTERING DATA

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Abstract. The problem of recovering the asymptotics of a short range perturbation of the Euclidean Laplacian on $\mathbb{R}^n$ from fixed energy scattering data is studied. It is shown that for $n \geq 3$ the asymptotics of a magnetic potential are determined, modulo Gauge invariance, by its scattering matrix at a fixed non-zero energy. This result also holds for a wide class of scattering manifolds.

1. Introduction

In this paper, we examine the question of the recovery of a magnetic potential, for a fixed short range perturbation of the Euclidean metric, from fixed energy scattering data. We show that, modulo a Gauge invariance, the asymptotics of the magnetic potential are determined. The analogous results for the recovery of asymptotics of metrics and electric potentials have been obtained in [6] and [7]. Our approach is to use the techniques of [6] and [7] and invert the arising integral transforms. Although it is clear our results hold in a general scattering manifold, with boundary the unit sphere, $\mathbb{S}^{n-1}$, or a torus equipped with the Euclidean metric, the real interest in this result is in the case of perturbations of the Euclidean Laplacian, and since the proof in the general case adds little other than obscurity, we will restrict ourselves to Euclidean space.

The problem of determining a magnetic potential in $\mathbb{R}^3$, with the Euclidean metric, from the scattering matrix has been considered by several authors, see for example [1], [3], [13], [14], [15], [16], and references cited there. It was shown in [13] that for compactly supported electric potential V and magnetic potential A, both V and the magnetic field $\nabla \times A$, are determined uniquely by the scattering matrix at one energy. In [1] these results were extended to exponentially decaying potentials.

The results proven here in particular give that if a magnetic potential A is a classical symbol of order -2, the asymptotic expansion of $\nabla \times A$ is determined by the full symbol of the scattering matrix. This raises the interesting question of determining the sharp rate of decay that guarantees uniqueness of the magnetic field. For the case of an electric potential, the results of Grinevich and Novikov [2] show that, in dimension two, there exist rapidly decaying electric potentials which are not determined by their scattering matrices. Examples in higher dimensions or for a magnetic field are unknown to us.

We study short range perturbations of Euclidean space. So we assume g_{ij} is a metric on $\mathbb{R}^n$, $n \geq 3$, with $g_{ij} = \delta_{ij} + l_{ij}$ and l_{ij} a (real-valued,) classical symbol of order -2. We let G denote the determinant of g_{ij} and we use g^{ij} to denote the pointwise matrix inverse. We take a magnetic potential $(A_1, \dots, A_n)$ and an electric potential V also to be real-valued classical symbols of order -2. Let P be the operator,

$$P = G^{-1/4} \sum_{j,k=1}^{n} \left(i\frac{\partial}{\partial x_j} + A_j \right) G^{1/2} g^{jk} \left(i\frac{\partial}{\partial x_k} + A_k \right) G^{-1/4} + V \tag{1.1}$$

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The Poisson operator is then the map that takes $f \in C^\infty(S^{n-1})$ to the smooth function u such that $(P-\lambda^2)u = 0$, and

$$u = e^{i\lambda|x|}|x|^{-\frac{n-1}{2}} f(x/|x|) + e^{-i\lambda|x|}|x|^{-\frac{n-1}{2}} g(x/|x|) + O(|x|^{-\frac{n+1}{2}}).$$

The kernel of the Poisson operator will be a smooth function on $\mathbb{S}^{n-1} \times \mathbb{R}^n$ with singular asymptotics. The scattering matrix is the map on $C^\infty(\mathbb{S}^{n-1})$,

$$S(\lambda) : f \longmapsto g. \tag{1.2}$$

In Section 3 we will prove

Proposition 1.1. *With $g, (A_1, \dots, A_n)$ and V as above, we have that $a^*S(\lambda)$ is a zeroth order classical pseudo-differential operator, where $a(\omega) = -\omega$.*

We remark that saying that $a^*S(\lambda)$ is a zeroth order classical pseudo-differential operator is equivalent to saying that $S(\lambda)$ is a classical Fourier integral operator of order 0 associated with geodesic flow at time π. This proposition is clear from [9], see also [5] and [17], by just observing that the arguments are not changed by adding a first order short range self-adjoint perturbation, we therefore only present the parts of the proof which relate to the proof of our main result:

Theorem 1.1. *Let g and V be as above. Let $(A_1, \dots, A_n)$ and $(A_1^{'}, \dots, A_n^{'})$ be real-valued classical symbols of order -2 with $S(\lambda)$ and $S'(\lambda)$ the associated scattering matrices at energy $\lambda \neq 0$. If $A_j - A_j^{'}$ is of order $-k$ with $k \geq 2$ then $S(\lambda) - S'(\lambda)$ is of order $1-k$. If $\sum\limits_{j=1}^{n} (A_j - A_j^{'})dx_j = B$ with lead term $B^{(k)}$, and $B^{(k)}$ is aradial, then it induces a 1-form on $\mathbb{S}^{n-1}$ and the principal symbol of $S(\lambda) - S'(\lambda)$ determines and is determined by*

$$\int_0^\pi \langle B^{(k)}(\gamma(s)), \frac{d\gamma}{ds}(s)\rangle (\sin s)^{k-1} ds$$

for all geodesics γ on $\mathbb{S}^{n-1}$, where s is the arc-length of γ, and we regard $B^{(k)} = \sum B_j^{(k)} dx_j$ as a one form canonically pairing with the vector $\frac{d\gamma}{ds}(s)$.

We say a one-form is *aradial* if its pairing with the radial vector field, $x\frac{\partial}{\partial x}$, is zero.

We deduce from this that given two magnetic fields A_j, $j = 1, 2$, such that $A_1 - A_2$ is aradial and $A_1 - A_2 = |x|^{-k} B^{(k)}(\frac{x}{|x|}) + O(|x|^{-k-1})$, as $|x| \to \infty$, then we can recover a weighted integral of $B^{(k)}$ along lifted geodesics in the cosphere bundle of the boundary. Such X-ray transforms have previously arisen in [6] and [7] and also in the linearization of the problem of recovery of a metric from its hodograph- that is the question whether a manifold can be determined from the length of its geodesics, see for example [10, 11]. Applying one of the results of Michel [12] we obtain.

Theorem 1.2. *Let g and V be as above and let n be greater than or equal to 3. Let $(A_1, \dots, A_n)$ and $(A_1^{'}, \dots, A_n^{'})$ be real-valued classical symbols of order -2 and let $S(\lambda)$ and $S'(\lambda)$ denote the associated scattering matrices. If $S(\lambda) - S'(\lambda)$, $\lambda \neq 0$, is of order $-k$, then there exists $\phi_k \in C^\infty(\mathbb{R}^n)$, such that $A - A' - d\phi_k$ is a symbol of order $-k-1$. Applying Borel's lemma we deduce that if $S(\lambda) - S'(\lambda)$ is smoothing, there exists $\phi \in C^\infty(\mathbb{R}^n)$, such that $A - A' - d\phi$ is rapidly decaying.*

We remark that the methods used here can also be applied to show that the full symbols of the scattering matrix at three different energies determine the asymptotics of the metric, g_{ij}, modulo diffeomorphisms fixing infinity, the magnetic potential, modulo Gauge invariance, and an electric potential. To see this we simply observe that at the k^{th} level a metric perturbation will yield a principal symbol in the difference of the scattering matrices which is homogeneous in λ of order k, a magnetic potential perturbation will yield a principal symbol which is homogeneous of order $k-1$ and for a potential will yield one of order $k-2$. So one can recover all three from the principal symbol at three positive energies, and any two from the principal symbol at two energies.

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2. Gauge Invariance

Let P be the operator,

$$P = G^{-1/4} \sum_{j,k=1}^{n} \left(i\frac{\partial}{\partial x_j} + A_j \right) G^{1/2} g^{jk} \left(i\frac{\partial}{\partial x_k} + A_k \right) G^{-1/4} + V. \tag{2.1}$$

We observe that if u is an eigenfunction for P then $w = e^{-i\phi}u$ is an eigenfunction for the operator obtained by adding $d\phi$ to A, where A is the one form $\sum A_j dx_j$. So if ϕ is a classical symbol of order -1 then u, w have the same lead terms in their asymptotics and thus we conclude the operators have the same scattering matrices.

Proposition 2.1. *Given a vector-valued function $A = (A_1, ..., A_n)$ on $\mathbb{R}^n$ such that A_i is a classical symbol of order -2, there exists a classical symbol, ϕ, of order -1 such that*

$$\sum_j x_j \left(A_j - \frac{\partial \phi}{\partial x_j} \right) \in \mathcal{S}(\mathbb{R}^n). \tag{2.2}$$

Proof. We can express (2.2) as

$$r\frac{\partial \phi}{\partial r} - \langle x, A\rangle = O\left(r^{-\infty}\right), \quad r = |x|. \tag{2.3}$$

Since $\langle x, A\rangle$ is a classical symbol of order -1, there exist $a_m \in C^\infty\left(\mathbb{S}^{n-1}\right)$, $m = 1, 2, ...$, such that $\langle x, A\rangle \sim \sum_{m=1}^\infty a_m(\omega) r^{-m}$, $\omega = \frac{x}{|x|}$. Since $r\frac{\partial}{\partial r} r^{-m} = -mr^{-m}$ we can use Borel's lemma to construct $\phi \in C^\infty(\mathbb{R}^n)$ such that $\phi(x) \sim \sum_{j=1}^\infty \frac{-1}{m} a_m(\omega) r^{-m}$. It is then clear that ϕ satisfies (2.3). $\square$

Thus we conclude that every magnetic potential is Gauge equivalent to one which is aradial modulo Schwartz. It is clear from Euler's relation that this potential is unique modulo Schwartz.

We remark that if one wants to recover a metric and a magnetic field, first one first constructs a diffeomorphism to put the metric into a model form, and then a Gauge equivalence to make the magnetic potential aradial.

3. Computing the Symbol

In this section, we compute the symbol of the scattering matrix and prove Theorem 1.1. In particular given two magnetic fields which agree to some order, we compute the principal symbol of the difference of the associated scattering matrices in terms of the lead term of the difference of the magnetic fields. We use the techniques of [9], [6] and [5] to construct the Poisson operator and compute the symbols. We proceed explicitly where possible but occasionally fall back on results from [9] for brevity.

Following the ideas of [9], [5], [6], we look to construct the Poisson operator for the problem as a sum of oscillatory integrals and then use this to read off the properties of the scattering matrix.

In particular, we attempt to construct the symbol of the Poisson operator as

$$e^{i\lambda x.\omega}(1+b(x,\omega)), \tag{3.1}$$

with b a classical symbol of order -1 in x and smooth in ω. We will see that this ansatz works away from the set $\omega = -x/|x|$. Using the assumptions on the metric G_{ij} and applying $P-\lambda^2$, we obtain

$$e^{i\lambda x.\omega}\left(-2i\lambda\omega.\frac{\partial b}{\partial x} + -2\lambda\sum_{j=1}^{n}\omega_j A_j + V + R.\right), \tag{3.2}$$

where R is a symbol of -3. More generally, if c is a symbol of order $-k \geq 2$ and we apply $P-\lambda^2$, to $e^{i\lambda x.\omega}c$, we get

$$e^{i\lambda x.\omega}\left(-2i\lambda\omega.\frac{\partial c}{\partial x} + e\right), \tag{3.3}$$

with e a symbol of order $-k-2$. So when we wish to iteratively solve away an error we need to solve at each stage an equation of the form

$$-2i\lambda\omega.\frac{\partial c}{\partial x} = d$$

with c one order higher than d. That we can solve for c smoothly to infinite order in a neighbourhood of $x/|x| = \omega$ is just a repetition of the argument in the proof of Proposition 18 of [9].

We let b have asymptotic expansion $\sum_{j=1}^{\infty} b_{-j}$, let A_l have expansion $\sum_{j=2}^{\infty} A_l^{(-j)}$ and V have expansion $\sum_{j=2}^{\infty} V_{-j}$. In order to compute b, we observe that the lead term in (3.2) is

$$-2i\lambda\omega.\frac{\partial b_1}{\partial x} - \lambda\sum_{l=1}^{n}\omega_l A_l^{(-2)} + V_{-2}.$$

We want this to be zero.

Note as these are homogeneous functions this is really an equation on the sphere. We therefore take coordinates (r,s,θ) where $r=|x|$, s is the geodesic distance (in the unit sphere) of $x/|x|$ from ω and θ is the angular coordinate about ω. Note the coordinate system depends on ω but we shall suppress ω in our notation most of the time.

Now without loss of generality, we can take ω to be the north pole. Then

$$\omega.\partial_x = \partial_{x_n}.$$

Now $\cos(s) = \frac{x_n}{|x|}$, $r=|x|$. The θ coordinate will be purely parametric. We have

$$\frac{\partial s}{\partial x_n} = \frac{-1}{\sqrt{1-\frac{x_n^2}{|x|^2}}}\left(\frac{1}{|x|} - \frac{x_n^2}{|x|^3}\right) = -r^{-1}\sin(s),$$

and

$$\frac{\partial r}{\partial x_n} = \cos(s).$$

So applying $\omega.\partial_x$ to $b_{-j}(s,\theta;\omega)r^{-j}$, we obtain

$$r^{-j-1}\left[-\sin(s)\frac{\partial}{\partial s} - j\cos(s)\right]b_{-j}.$$

Let $W_{-2} = -\lambda \sum_{j=1}^{n} \omega_j A_j^{(-2)} - V_{-2}$. So for $j = 1$, we have taking $r = 1$

$$2i\lambda(\sin(s)\partial_s + \cos(s))b_{-1} = W_{-2},$$

which is equivalent to

$$2i\lambda\partial_s(\sin(s)b_{-1}) = W_{-2}.$$

We want b to be smooth at $s = 0$, so

$$\sin(s)b_{-1} = \frac{1}{2i\lambda}\int_0^s W_{-2}ds',$$

which implies that

$$b_{-1}(s,\theta;\omega) = \frac{i}{2\lambda\sin(s)}\int_0^s W_{-2}(s',\theta;\omega)ds'. \tag{3.4}$$

This will be singular as $s \to \pi-$, but let's ignore that for now. Now suppose we have chosen the first j terms so we have an error $d \in S_{cl}^{-j-1}$ with lead term $d_{-j-1}(s,\theta;\omega)r^{-j-1}$. We then want to solve the transport equation,

$$-2i\lambda\omega.\partial_z(b_{-j}jr^{-j}) + d_{-j-1}r^{-j-1} = 0,$$

as above we get

$$2i\lambda[\sin(s)\partial_s + j\cos(s)]b_{-j} + d_{-j-1} = 0.$$

We solve this to obtain,

$$b_{-j}(s,\theta,\omega) = \frac{i}{2\lambda(\sin(s))^j}\int_0^s (\sin(s'))^{j-1}d_{-j-1}(s',\theta;\omega)ds'. \tag{3.5}$$

So away from $s = \pi$, we can achieve an error in $S^{-\infty}$ by applying Borel's lemma. ie away from the antipodal point. Note that we have a focusing of the geodesics and as well as the fact the solutions blow-up we also have that they will have different values according to the angle. In particular provided d_{-j-1} does not grow faster than $(\pi - s)^{1-j}$ we have that b_{-j} grows as $(\pi - s)^{-j}$.

Before introducing a second ansatz to cope with the antipodal point, we compare the Poisson operators associated to two different magnetic potentials. Suppose $(A_1, \ldots, A_n)$ and $(A_1^{'}, \ldots, A_n^{'})$ are both classical symbols of order -2 and the difference is $(B_1, \ldots, B_n)$ which is a classical symbol of order $-k$, with lead term $(B_1^{(-k)}, \ldots, B_n^{(-k)})$. If we keep the metric and potential fixed then the first $k-2$ forcing terms above are unchanged and the forcing terms at level $k-1$ will differ by $W_{-k} = -\lambda \sum_{j=1}^{n} \omega_j B_j^{(-k)}$. (Note that the change in the zeroth order term will be lower order.)

Thus the lead term of the difference of the Poisson operators will be

$$\frac{ir^{1-k}}{2\lambda(\sin s)^{k-1}}\int_0^s W_{-k}(s',\theta;\omega)(\sin s')^{k-2}ds'. \tag{3.6}$$

This is the important result in our construction as we will see that the lead term of this as $s \to \pi-$ is essentially the principal symbol of the difference of the scattering matrices. We therefore want an invariant interpretation of

$$\int_0^\pi (\sin s)^{k-2}W_{-k}(s,\theta;\omega)ds.$$

If we take ω to be the north pole and rotate so that $\theta' = (1, 0, \ldots, 0)$, the computation lies entirely in the (x_1, x_n) plane and $W_k(s)$ equals $-\lambda B_n^{(-k)}(s)$. Now if we assume B is aradial then

$$-\left\langle B^{(-k)}(\gamma(s)), \frac{d\gamma}{ds}(s)\right\rangle \sin s = B_n^{(-k)}.$$

To see this, with out loss of generality, we can take $\gamma(s) = (0, \ldots, 0, \sin s, \cos s)$. We then have $\sin(s)B_{n-1}^{(-k)} + \cos(s)B_n^{(-k)} = 0$ and have $\gamma'(s) = (0, \ldots, 0, \cos(s), -\sin(s))$. The equivalence then follows from a simple manipulation. So by rotational invariance we deduce that in general the lead term of the difference of the Poisson operators is

$$\frac{ir^{1-k}}{2(\sin s)^{k-1}} \int_0^s \left\langle B^{(-k)}(\gamma(s')), \frac{d\gamma}{ds'}(s')\right\rangle (\sin s')^{k-1} ds' \tag{3.7}$$

and that the lead singularity as $s \to \pi-$ is

$$\frac{ir^{1-k}}{2(\pi - s)^{k-1}} \int_0^\pi \langle B^{(-k)}(\gamma(s')), \frac{d\gamma}{ds'}(s')\rangle (\sin s')^{k-1} ds'. \tag{3.8}$$

The remainder of the construction of the Poisson operator and the computation of the symbol is now just a repetition of the arguments in [9] or [5]. We sketch these for completeness.

Taking ω to be the north pole, close to the south pole we look for the Poisson operator in the form,

$$\int_0^\infty \int_{\mathbb{S}^{n-2}} \left(\frac{1}{S|x|}\right)^\gamma S^\alpha e^{i(Sx'.\mu - \sqrt{1+S^2}|x|)} a\left(\frac{1}{S|x|}, S, \mu\right) dS d\mu, \tag{3.9}$$

where $x = (x', x_n)$ and $a(t, S, \mu)$ a smooth function compactly supported on $[0, \epsilon) \times [0, \epsilon) \times \mathbb{S}^{n-2}$ and $\alpha = \frac{n-3}{2}, \gamma = -\frac{n-1}{2}$. We assume that ω has been rotated to the north pole. Note that for $|x|$ in a compact set the integral is supported on a compact set and so we have no problems with convergence - in particular the integral yields a smooth function.

This can be seen as introduction of polar coordinates around the south pole. In the lower hemi-sphere, away from the south pole, where the introduction of polar coordinates is a diffeomorphism, this ansatz is equivalent to the original one - this follows from an application of stationary phase (see [5]). However the lead term in $|x|$ at order $-k$ is now allowed to be singular of order $-k$ as $s \to \pi-$ (which corresponds to $S = 0+$) and there is no constraint on the values for different angles matching. This allows the transport equations to be solved right up to the antipodal point and to all orders. This gives the Poisson operator up to an error. The precise regularity of this error is established in Section 3 of [5]. To obtain the Poisson operator one can remove the error by applying the resolvent which yields a term of the form $e^{-i\lambda|x|}|x|^{-\frac{n-1}{2}}h(x)$ with h a classical zeroth order symbol - (see Section 3 of [5]) this term will not affect the singularities in the distributional asymptotics of the Poisson operator.

We recall Proposition 3.4 of [5], which is a special case of Proposition 16 of [9].

Proposition 3.1. *If $u(x, \omega)$is of the form (3.9) then $e^{i\lambda|x|} \int u(|x|\theta, \omega) f(\theta, \omega) d\theta d\omega$ is a smooth symbolic function in $|x|$ of order $-1 - \alpha$ and its lead coefficient is $|x|^{-\alpha-1}\langle K, f\rangle$ where K is the pull-back of the Schwartz kernel of a pseudo-differential operator of order $\alpha - \gamma - (n-2)$ by the map $\theta \mapsto -\theta$. The principal symbol of K determines and is determined by the lead term of the symbol, $a(t, S, \mu)$, of u as $S \to 0+$.*

The fact that the scattering matrix is the pull-back of a pseudo-differential operator is now immediate. To deduce Theorem 1.1, we observe that from our computations above, and by the assumption that $A_j - A_j'$

is of order $-k$, we have that the difference of the second ansatzs for the two Poisson operators, i.e the construction near the south pole, will be of the same form but with α increased by $k-1$ and the lead term of the symbol of the difference of the Poisson operators as $S \to 0+$ will be a constant multiple of

$$S^{n-k-1} \int_0^{\pi} \langle B^{(-k)}(\gamma(s')), \frac{d\gamma}{ds'}(s') \rangle (\sin s')^{k-1} ds'.$$

So Theorem 1.1 then follows from Proposition 3.1.

4. Proof of Theorem 1.2

From Proposition 2.1 we may assume that

$$A - A' = \sum_{j=1}^{n} \left(A_j - A_j^{'} \right) dx_j, \; A_j - A_j^{'} = |x|^{-k} B_j^{(k)} \left(\frac{x}{|x|} \right) + O\left(|x|^{-k-1} \right),$$

$$k \geq 2, \text{ as } |x| \to \infty, \quad \text{with} \quad \sum_{j=1}^{n} B_j^{(k)} \left(\frac{x}{|x|} \right) x_j = 0.$$

Thus $B^{(k)}$ defines a 1-form on $\mathbb{S}^{n-1}$. As in [6] and [7] we begin by defining

$$I_m^k(\alpha) = \int_0^{\pi} B^{(k)} \left(\gamma(s+\alpha), \frac{d}{ds}\gamma(s+\alpha) \right) \sin^{m-1}(s) \, ds,$$

where γ is a fixed geodesic on $\mathbb{S}^{n-1}$. If $m \geq 3$, find that

$$\frac{d^2 I_m^k}{d\alpha^2}(\alpha) + (m-1)^2 I_m^k(\alpha) = (m-1)(m-2) I_{m-2}^k(\alpha).$$

So we find that if $I_m^k = 0$, then $I_{m-2}^k = 0$. We know from Theorem 1.1 that $I_k^k(\alpha) = 0$, $k \geq 2$, for every $\alpha \in \mathbb{R}$. Thus it follows that for every geodesic γ,

$$\begin{aligned} I_1^k(0) &= \int_0^{\pi} B^{(k)} \left(\gamma(s), \frac{d}{ds}\gamma(s) \right) ds = 0, \text{ if } \quad k \quad \text{is odd}, \\ I_2^k(0) &= \int_0^{\pi} B^{(k)} \left(\gamma(s), \frac{d}{ds}\gamma(s) \right) \sin(s) \, ds = 0, \text{ if } \quad k \quad \text{is even}. \end{aligned} \tag{4.1}$$

Differentiating again, we deduce that $B^{(k)}$ is of opposite parity to k. Since $\sin(s)$ is a fixed multiple of the coordinate x_1 of $\gamma(s)$, we deduce, by the rotational invariance of $\mathbb{S}^{n-1}$ and the second equation of (4.1), that

$$\int_0^{\pi} T(\gamma(s)) B^{(k)} \left(\gamma(s+\alpha), \frac{d}{ds}\gamma(s+\alpha) \right) ds = 0, \quad 1 \leq j \leq n, \quad \text{if} \quad k \quad \text{is even}, \tag{4.2}$$

where T is any homogeneous linear function on $\mathbb{R}^n$. It is immediate from Theorem 1.7 of [12] that if a 1-form ω defined on $\mathbb{S}^{n-1}$ satisfies

$$\int_0^{2\pi} \omega(\gamma(s), \gamma'(s)) ds = 0,$$

for every geodesic γ, then ω is equal to an exact form plus an odd form, as a 1-form on the sphere. Since we already know that $TB^{(k)}$ is even we deduce that is exact and therefore closed. We now show B vanishes at the north pole. Since everything is rotationally invariant this suffices to prove B is zero everywhere.

We can take coordinates on the sphere simply by projecting onto the first $n-1$ coordinates. We then have for $j = 1, \ldots, n-1$, that

$$d(x_j B^{(k)}) = 0.$$

Evaluating at $x_j = 0$, for each j we have that at the north pole,

$$dx_j \wedge B^{(k)} = 0$$

for each j and thus that $B^{(k)}$ is zero there and we are done.

If k is odd, we have that upon differentiating that $B^{(k)}$ is even and we also have that the integral around any closed loop times any homogeneous polynomial of order two is zero. Applying the arguments above to $x_n B^{(k)}$, we deduce that $x_n B^{(k)}$ is zero at the north pole and Theorem 1.2 follows.

References

[1] G. Eskin, J. Ralston, *Inverse Scattering Problem for the Schrödinger Equation with Magnetic Potential at a Fixed Energy.* Comm. Math. Phys. **173**, 199-224 (1995)

[2] P.G. Grinevich, R.G. Novikov, *Transparent Potentials at Fixed Energy in Dimension Two. Fixed Energy Dispersion Relations For The Fast Decaying Potentials.* Comm. Math. Phys. **1174**, 409-446.

[3] G. Henkin and R. Novikov, *The $\overline{\partial}$-Equation in the Multidimenional Inverse Scattering Problem.* Russian Math. Surveys **42** (1987), no.3 109-180.

[4] M.S. Joshi, *Recovering Asymptotics of Coulomb-like Potentials.* S.I.A.M. Journal of Mathematical Analysis **Vol. 30 No 3**, 516-526, (1999)

[5] M.S. Joshi, *Explicitly Recovering Asymptotics of Short Range Potentials.* To appear in Comm. in P.D.E

[6] M.S. Joshi, A. Sá Barreto, *Recovering Asymptotics of Short Range Potentials.* Comm. Math. Phys. **193**, 197-208 (1998)

[7] M.S Joshi, A. Sá Barreto, *Recovering Asymptotics of Metrics from Fixed Energy Scattering Data.* to appear in *Invent. Math.*

[8] R.B. Melrose, *Spectral and Scattering Theory for the Laplacian on Asymptotically Euclidean spaces.* (M. Ikawa, ed.) Marcel Dekker, 1994.

[9] R.B. Melrose, M. Zworski, *Scattering Metrics and Geodesic Flow at Infinity.* Invent. Math. 124, 389-436 (1996).

[10] R. Michel, *Problèmes d'Analyse Géométrique Liés a La Conjecture de Blaschke.* Bull. So. math. France 101, p17-69 (1973).

[11] R. Michel, *Sur Certains Tenseurs Symétriques des Projectifs Réels.* J. Math. pures et appl. 51, 273-293 (1972).

[12] R. Michel, *Sur Quelques Probèmes de Géométrie Globale Des Geodesiques.* Bol. Soc. Bras. Mat. **9**, no.2 (1978), 19-38.

[13] G. Nakamura, G. Uhlmann and Z. Sun, *Global Identifiability For an Inverse Problem For the Schr̈odinger Equation with Magnetic Field.* Math. Annal. **303** (1995), no.3, 377-388.

[14] F. Nicoleau, *A Stationary Approach to Inverse Scattering for Schrödinger Operators With First Order Perturbation.* Comm. Partial Diff. Equations **22** (1997), no. 3-4, 527-553.

[15] R. Novikov, *The Inverse Scattering Problem on a Fixed Energy Level for the Two Dimensional Schrödinger Operator.* J. of Func. Anal. **103** (1992), no. 2, 409-463.

[16] T. Shiota, *An Inverse Problem for the Wave Equation With First Order Perturbation.* Amer. J. Math. **107** (1985), no.1 241-251.

[17] A. Vasy, *Geometric Scattering Theory for Long-Range Potentials and Metrics.* I.M.R.N. 1998 no 6, 285-315

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