Contemporary Mathematics

Radiation Fields and Inverse Scattering on Asymptotically Euclidean Manifolds

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1. Introduction and Statement of the Results

Geometric scattering theory has been intensely studied in recent years, see for example [**12**] for a survey. In several cases of complete metrics in the interiors of compact manifolds, Melrose, [**12**] and references cited there, has developed the corresponding scattering theory by carefully analyzing the structure of the solutions to the Schrödinger equation in a neighborhood of infinity. Here we consider the asymptotically Euclidean case, which are smooth compact manifolds with boundary equipped with a complete metric that resembles the Euclidean space near the boundary. This case has been studied by Melrose [**13**], Melrose and Zworski [**14**] and Vasy [**19**]. See also [**12**] for other references. We use the radiation fields, introduced by Friedlander [**3, 4**], to develop the scattering theory in the asymptotically Euclidean case from a dynamical point of view. We then use this new approach to study the inverse problem of determining the manifold and the metric from the scattering matrix at *all energies*.

The Euclidean space model: Let $\mathbb{S}^n_+$ denote the upper hemisphere of the unit sphere in $\mathbb{R}^{n+1}$, i.e.

$$\mathbb{S}^n_+ = \{z \in \mathbb{R}^{n+1} : |z| = 1, \quad y_1 > 0\}.$$

Its boundary is $\partial\mathbb{S}^n_+ = \mathbb{S}^{n-1}$.

Consider the following compactification of the Euclidean space

$$SP : \mathbb{R}^n \longrightarrow \mathbb{S}^n_+$$
$$z \longmapsto \left(\frac{1}{(1+|z|^2)^{\frac{1}{2}}}, \frac{z}{(1+|z|^2)^{\frac{1}{2}}}\right) = (x, y).$$

Notice that

$$x = \frac{1}{(1+|z|^2)^{\frac{1}{2}}}$$

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is a defining function of $\partial\mathbb{S}^n_+$, i.e.

$$x^{-1}(0) = \partial\mathbb{S}^n_+, \quad \text{and} \quad dx \neq 0 \quad \text{when} \quad x = 0.$$

The Euclidean metric when pushed forward to $\mathbb{S}^n_+$ has the form

$$g = \frac{dx^2}{x^4} + \frac{\mathcal{H}}{x^2}, \tag{1.1}$$

Here $\mathcal{H}$ is a symmetric tensor which restricts to the standard metric on $\mathbb{S}^{n-1}$.

Moreover, as pointed out in [**14**], any perturbation of the Euclidean metric which behaves like

$$g_{ij} = \delta_{ij} + \frac{1}{|z|^2} h_{ij}\left(\frac{1}{|z|}, \frac{z}{|z|}\right), \quad \text{as } |z| \to \infty$$

satisfies (1.1) when pushed forward to $\mathbb{S}^n_+$.

As defined by Melrose [**13**], a smooth compact manifold X with boundary, ∂X, is called asymptotically Euclidean, or scattering manifold, when it is equipped with a Riemannian metric g, which is smooth in the interior of X, denoted by $\overset{\circ}{X}$, and can be written as (1.1) near ∂X, where $x \in C^\infty(X)$ is a smooth defining function of ∂X and $\mathcal{H}$ is a symmetric tensor which restricts to a smooth Riemannian metric h_0 on ∂X.

Remark 1: As shown in [**13**], once (1.1) is known to exist, it determines x up to terms vanishing to second order at ∂X, and hence it determines

$$h_0 = \mathcal{H}|_{\partial X}. \tag{1.2}$$

We will denote the dimension of X by n, and therefore ∂X has dimension $n-1$.

Remark 2: It was proved by Joshi and Sá Barreto [**8**] that if g satisfies (1.1) there exists $\epsilon > 0$ and a product structure $X \sim \partial X \times [0, \epsilon)$ in which

$$g = \frac{dx^2}{x^4} + \frac{h(x, y, dy)}{x^2}. \tag{1.3}$$

These are the analogue of boundary normal coordinates on a compact manifold with boundary.

We will fix such a decomposition, and from now on $x \in C^\infty(X)$ will be as in (1.3). Let Δ be the (positive) Laplace operator with respect to the metric g.

Melrose has shown that given $f \in C^\infty(\partial X)$ and $\lambda \in \mathbb{R}$, $\lambda \neq 0$, *there exists a unique*
$u \in C^\infty(\overset{\circ}{X})$ satisfying

$$\begin{gathered} (\Delta - \lambda^2)u = 0 \text{ in } \overset{\circ}{X}, \\ u = x^{\frac{n-1}{2}} e^{i\frac{\lambda}{x}} F + x^{\frac{n-1}{2}} e^{-i\frac{\lambda}{x}} G, \\ F,\ G \in C^\infty(\overline{X}), \quad F|_{\partial X} = f. \end{gathered} \tag{1.4}$$

This leads to *the stationary definition* of the scattering matrix at energy λ as the operator

$$\begin{gathered} A(\lambda) : C^\infty(\partial X) \longrightarrow C^\infty(\partial X) \\ f \longmapsto G|_{\partial X}. \end{gathered} \tag{1.5}$$

Melrose and Zworski showed that $A(\lambda)$ is a classical Fourier integral operator of order zero associated to the geodesic flow at time π given by the metric h_0 on ∂X.

Here we address the following question on inverse scattering:

THEOREM 1.1. *What information about (X, g) can be recovered from $A(\lambda)$ for fixed $\lambda \in \mathbb{R}$, or for all $\lambda \in \mathbb{R}$?*

For λ fixed we have

THEOREM 1.1. [**8**] *If ∂X is either a sphere of radius 1, the real projective space or a sphere of irrational radius, and if $h(0, y, dy)$ and $D_x h(0, y, dy)$ are known, then the full symbol of $A(\lambda)$, for λ fixed, determines $D_x^k h(x, y, dx)$ at $x = 0$ for $k \geq 2$. It does not necessarily determine $h(0, y, dy)$ or $D_x h(0, y, dy)$.*

This is related to a result of Lee and Uhlmann [**10**] which states that the (stationary) Dirichlet-to-Neumann map determines the Taylor series of a smooth Riemannian metric on a compact manifold with boundary.

The basic method of the proof, as in the case of Lee and Uhlmann, is to compute the symbol of the scattering matrix and show that it determines the the coefficients of the tensor h. However in this case $A(\lambda)$ is a Fourier integral operator and one does not recover the coefficients of h directly, but only their integral along certain geodesics. This leads to extra difficulties and one cannot recover the coefficients from those integrals if the boundary is arbitrary, hence the assumption on the boundary made in Theorem 1.1.

The purpose of this paper is to contribute to the answer of the question in the case where $A(\lambda)$ is known for all $\lambda \neq 0$. We observe that symbolic computations leave out all the information contained in the smoothing part of the operator. Moreover, the dependence on λ of the scattering matrix is very complicated and it is hard to obtain much information about the manifold X and the metric g from $A(\lambda)$. So it is necessary to look at another approach. As shown by Uhlmann [**18**] for a compactly supported perturbation of the Euclidean metric, the problem of recovering the metric (up to isometries) from the scattering matrix at all energies is the equivalent to the question of determining the metric (also up to isometries) in a ball containing the support of the perturbation from the Dirichlet-to-Neumann map for the wave equation. It is a theorem of Belishev and Belishev- Kurylev [**1, 2**] that the wave Dirichlet-to-Neumann map determines the metric.

Here we try to use the method of [**1, 2**], see also [**9**] to study question of determining the metric and the manifold from $A(\lambda)$ for all $\lambda \neq 0$. For that we need the analogue of the Dirichlet-to-Neumann map. This rôle is played by the radiation fields, which were introduced by F.G. Friedlander [**3, 4**]. We will then use the radiation fields to arrive at an equivalent definition of the scattering matrix.

The method [**1, 2**] relies on the boundary control method. This is substituted here by a control method for radiation fields.

Friedlander proved [**3, 4**]

THEOREM 1.2. *For $f_1, f_2 \in C_0^\infty(X)$ compactly supported in the interior of X, let*

$u(t,z) \in C^\infty(\mathbb{R}^+ \times X)$ satisfy

$$\begin{aligned} \left(D_t^2 - \Delta\right) u(t,z) = 0, \ \text{on } \mathbb{R}\times \overset{\circ}{X}, \\ u(0,z) = f_1(z), \quad D_t u(0,z) = f_2(z). \end{aligned} \tag{1.6}$$

Let $z = (x,y) \in (0,\epsilon) \times \partial X$ be local coordinates near ∂X in which (1.3) hold. Let $H(t) = 1$ for $t > 0$ and $H(t) = 0$ otherwise, denote the Heaviside function. Then there exist $w_k \in C^\infty(\mathbb{R} \times \partial X)$, such that

$$x^{-\frac{n-1}{2}} (Hu)(s + \frac{1}{x}, x, y) \sim \sum_{k=0}^{\infty} x^k w_k(s,y), \quad \text{as } x \to 0. \tag{1.7}$$

In particular,

$$x^{-\frac{n-1}{2}} (Hu)(s + \frac{1}{x}, x, y)|_{x=0} = w_0(s,y),$$

is well defined.

Theorem 1.2 defines a map

$$\begin{aligned} \mathcal{R}_+ : C_0^\infty(\overset{\circ}{X}) \times C_0^\infty(\overset{\circ}{X}) \longrightarrow \ C^\infty(\mathbb{R} \times \partial X) \\ \mathcal{R}_+(f_1,f_2)(s,y) = \\ (x^{-\frac{n-1}{2}} D_t H u)(s + \frac{1}{x}, x, y)|_{x=0} = D_s w_0(s,y), \end{aligned} \tag{1.8}$$

which will be called the *forward radiation field.*

It is shown by Friedlander and Lax-Phillips that, in the odd dimensional Euclidean space $\mathbb{R}^n$,

$$\mathcal{R}_+(0,f)(s,\omega) = \left(\frac{\partial}{\partial s}\right)^{\frac{n-1}{2}} Rf(s,\omega),$$

where

$$Rf(s,\omega) = \int_{\langle x,\omega\rangle = s} f(x) \ \ dH_x$$

is the Radon transform. Thus the radiation field can be viewed as a generalization of the Radon transform.

Similarly one can prove that if $H_-(t) = H(-t)$

$$\lim_{x\to 0} (x^{-\frac{n-1}{2}} H_- u)(s - \frac{1}{x}, x, y) = w_0^-(s,y)$$

exists, and thus define the *backward radiation field* as

$$\begin{aligned} \mathcal{R}_- : C_0^\infty(\overset{\circ}{X}) \times C_0^\infty(\overset{\circ}{X}) \longrightarrow \ C^\infty(\mathbb{R} \times \partial X) \\ \mathcal{R}_-(f_1,f_2)(s,y) = \\ (x^{-\frac{n-1}{2}} D_t H_- u_-)(s - \frac{1}{x}, x, y)|_{x=0} = D_s w_0^-(s,y). \end{aligned} \tag{1.9}$$

Our main result is:

THEOREM 1.3. *If two asymptotically Euclidean manifolds,* (X_1, g_1) *and* (X_2, g_2) *satisfy*
(h1) $\partial X_1 = \partial X_2 = M$,
(h2) If $h_{0,i}$ $i = 1, 2$, *are the metrics on* M *defined by* (1.2), *then* $h_{0,1} = h_{0,2}$.
(h3) $A_1(\lambda) = A_2(\lambda)$, *as operators defined by* (1.5), *for all* $\lambda \in \mathbb{R} \setminus 0$,
then the following are equivalent:
(i) There exists a smooth diffeomorphism $\Psi : X_1 \longrightarrow X_2$ *such that*

$$\Psi = \mathrm{Id} \ \text{ at } M \quad \text{and} \quad \Psi^* g_2 = g_1.$$

(ii) There exists a boundary defining function x *and* $\epsilon_0 > 0$ *such that*

$$g_i = \frac{dx^2}{x^4} + \frac{h_i(x, y, dy)}{x^2}, \quad i = 1, 2, \quad x < \epsilon_0,$$

and such that the ranges

$$R_i = \{\mathcal{R}_{i,+}(0, f), \quad f \in C_0^\infty(\overset{\circ}{X}), \quad \mathrm{supp}(f) \subset \{x < \epsilon_0\}\}, \quad i = 1, 2,$$

are equal.

2. The Radiation Fields And The Scattering Matrix

For $w_0, w_1 \in C_0^\infty(\overset{\circ}{X})$ the energy norm of $w = (w_0, w_1)$ is defined by

$$||w||_E = \frac{1}{2} \int_X \left(|dw_0|_g^2 + |w_1|^2 \right) \, d\,\mathrm{vol}_g, \tag{2.1}$$

where $|dw_0|_g^2$ denotes the length of the co-vector with respect to the metric induced by g on T^*X, and define $H_E(X)$ as the closure of $C_0^\infty(\overset{\circ}{X}) \times C_0^\infty(\overset{\circ}{X})$ with the norm (2.1).

Let $W(t)$ be the map defined by

$$\begin{gathered} W(t) : C_0^\infty(\overset{\circ}{X}) \times C_0^\infty(\overset{\circ}{X}) \longrightarrow C_0^\infty(\overset{\circ}{X}) \times C_0^\infty(\overset{\circ}{X}), \\ W(t)\,(f_1, f_2) = (u(t, z), D_t u(t, z))\,, \quad t \in \mathbb{R}. \end{gathered} \tag{2.2}$$

The conservation of energy gives is a strongly continuous group of unitary operators.

By changing $t \leftrightarrow t - \tau$, the variable s then changes to $s \leftrightarrow s + \tau$ therefore $\mathcal{R}_\pm$ satisfy

$$\mathcal{R}_\pm \circ (W(\tau) f)\,(y, s) = \mathcal{R}_\pm f(y, s + \tau), \quad \tau \in \mathbb{R}. \tag{2.3}$$

So Theorem 1.2 shows that $\mathcal{R}_\pm$ are "twisted" translation representations of the group $W(t)$. That is, if one sets $\widetilde{\mathcal{R}_\pm}(f)(y, s) = \mathcal{R}_\pm f(y, -s)$, then

$$\widetilde{\mathcal{R}_\pm}(W(\tau)) = T_\tau \widetilde{\mathcal{R}_\pm}, \tag{2.4}$$

where T_τ denotes the right translation by τ in the s variable. So $\widetilde{\mathcal{R}_\pm}$ are translation representers in the sense of Lax and Phillips. Some results of Friedlander [**3, 4**] and Hassel and Vasy [**5**] are used in [**15**] to prove

THEOREM 2.1. *(*[**15**]*) The maps* $\mathcal{R}_\pm$ *extend to isometries*

$$\mathcal{R}_\pm : H_E(X) \longmapsto L^2(\partial X \times \mathbb{R}).$$

The *scattering operator* is defined to be the map

$$\mathcal{S} = \mathcal{R}_+ \circ \mathcal{R}_-^{-1}.$$

It is clearly a unitary in $L^2(\partial X \times \mathbb{R})$, and in view of (2.3), it commutes with translations in s. Therefore the Schwartz kernel $\mathcal{S}(s, y, s', y')$ of $\mathcal{S}$ is completely determined by its values at $s = s'$. In fact it satisfies

$$\mathcal{S}(s, y, s', y') = \mathcal{S}\left(\frac{s+s'}{2}, y, \frac{s+s'}{2}, y'\right).$$

The *scattering matrix* is defined by conjugating $\mathcal{S}$ with the partial Fourier transform in the s variable

$$\mathcal{A} = \mathcal{F}\mathcal{S}\mathcal{F}^{-1}.$$

$\mathcal{A}$ is a unitary operator in $L^2(\partial X \times \mathbb{R})$. Since $\mathcal{S}$ acts as a convolution in the variable s', if λ denotes the dual variable to s, $\mathcal{A}\mathcal{F}$ is a multiplication in λ, i.e

$$\mathcal{A}(\mathcal{F}F)(\lambda, y) = \int_{\partial X} \mathcal{A}(\lambda, y, y')\mathcal{F}F(\lambda, y')\, d\,\mathrm{vol}_{h_0}.$$

It is a consequence of the results Hassell and Vasy [**5**] that the stationary and dynamical definitions of the scattering matrix coincide. More precisely:

THEOREM 2.2. *([**15**]) The Schwartz kernel of the map $A(\lambda)$ defined by* (1.5) *is equal to $\mathcal{A}(\lambda, y, y')$.*

3. A support Theorem for $\mathcal{R}_+$

THEOREM 3.1. *([**15**]) Let $f \in C_0^\infty(\overset{\circ}{X})$ and let $s_0 < 0$. If $\mathcal{R}(0, f)(s, y) = 0$ (or $\mathcal{R}(f, 0) = 0$) for all $s < s_0$ then $f = 0$ in the set $\{(x, y) \in X : x < -\frac{1}{s_0}\}$.*

Theorem 3.1 is the generalization of the support theorem for the Radon transform, which is due to Helgason, and which says that if $f \in C_0^\infty(\mathbb{R}^n)$ and $Rf(s, \omega) = 0$ for $|s| > \rho$, then $f(z) = 0$ for $|z| > \rho$.

The proof relies on Hörmander's uniqueness theorem for the Cauchy problem across a strongly pseudoconvex surface, see theorem 28.3.4 of [**7**], and a generalization of due to Tataru [**17**]. We remark that this is the analogue of the control results needed in [**1, 2**]. However this is not enough to prove the full metric determination result, and we can only prove Theorem 1.3. To prove the full result we would need a support theorem for functions which are not *a priori* known to be compactly supported. As is well known, this is a delicate issue even in the Euclidean case, see [**6**].

4. Proof of Theorem 1.3

The implication $(i) \Rightarrow (ii)$ is obvious. We will prove that $(ii) \Rightarrow (i)$. So we assume that there exists a boundary defining function x and $\epsilon_0 > 0$ such that

$$g_i = \frac{dx^2}{x^4} + \frac{h_i(x, y, dy)}{x^2}, \quad i = 1, 2, \quad x < \epsilon_0$$

and such that the ranges

$$W = R_i = \{\mathcal{R}_{i,+}(0,f), \quad f \in C_0^\infty(\overset{\circ}{X}), \quad \text{supported in } x < \epsilon_0\}, \quad i = 1,2, \tag{4.1}$$

are equal. To do that we use the boundary control method of Belishev and Belishev & Kurylev [**1, 2**]. Let

$$\mathcal{M} = \{\mathcal{R}_{+,1}(0,f) : f \in L_c^2(\overset{\circ}{X}), \mathrm{supp}(f) \subset \{x < \epsilon_0\}\} =$$
$$\{\mathcal{R}_{+,2}(0,f) : f \in L_c^2(\overset{\circ}{X}), \mathrm{supp}(f) \subset \{x < \epsilon_0\}\}$$

and for $x_1 \in (0, \epsilon_0)$, let

$$\mathcal{M}_{x_1}^i = \{\mathcal{R}_{+,i}(0,f) : f \in L_c^2(\overset{\circ}{X}), \mathrm{supp}(f) \subset \{x_1 < x < \epsilon_0\}\}, i = 1,2.$$

It is clear that $\mathcal{M}_{x_1}^i$, $i = 1,2$, is a closed subspace of $\mathcal{M}$. It follows from the support theorem that

$$\mathcal{M}_{x_1} = \{F \in \mathcal{M} \text{ supported in } s \geq -\frac{1}{x_1}\}.$$

Let $f \in C_0^\infty(\overset{\circ}{X})$, with $\mathrm{supp}(f) \subset \{x < \epsilon_0\}$, let $x_1 \in (0, \epsilon)$ and H_{x_1} denote the Heaviside function and let

$$F(s,y) = \mathcal{R}_+(0,f) \quad \text{and } F_1(s,y) = \mathcal{R}_+(0, H_{x_1} f).$$

Finite speed of propagation gives that F_1 is supported in $s > -\frac{1}{x_1}$, and so this defines a map

$$\mathcal{P}_{x_1} : \mathcal{M} \longrightarrow \mathcal{M}_{x_1}.$$
$$F \longmapsto F_1$$

The discussion above can be translated into

Lemma 4.1. *$\mathcal{P}_{x_1} : \mathcal{M} \longrightarrow \mathcal{M}_{x_1}$ is the orthogonal projection and therefore independent of the metric.*

Now we analyze the singularities of the solution to the Cauchy problem (1.6) with $f_1 = 0$ and $f_2(z) = H_{x_1} f$, in coordinates (s,x,y). It is shown in [**15**] that (1.6) translates into

$$\begin{gathered} \left(\frac{\partial}{\partial x}\left(2\frac{\partial}{\partial s} + x^2\frac{\partial}{\partial x}\right) + A\frac{\partial}{\partial s} + Ax^2\frac{\partial}{\partial x} + \frac{n-1}{2}\left(\frac{3-n}{2} + xA\right)\right)u = 0 \\ u|_{s=-\frac{1}{x}} = 0, \qquad \frac{\partial u}{\partial s}|_{s=-\frac{1}{x}} = H(x_1 - x)f(x,y), \end{gathered} \tag{4.2}$$

where $A = \frac{1}{2}\frac{\partial \log|h|}{\partial x}$, and $|h|$ is the determinant of the tensor h, that is $|h| = \det(h_{ij})$. The solution to (4.2) is singular along two characteristic surfaces emanating from $\{s = -\frac{1}{x}, \quad x = x_1\}$, which are

$$\{s = -\frac{1}{x_1}\} \quad \text{and} \quad \{s = -\frac{2}{x} + \frac{1}{x_1}\}.$$

The solution u can then be expressed by

$$u(x,s,y) = F_1(x,s,y)H\left(s+\frac{1}{x_1}\right) + F_2(x,s,y)H\left(-s-\frac{2}{x}+\frac{1}{x_1}\right),$$
$$F_j \in C^\infty, \quad j=1,2.$$

We then compute the Taylor series of F_1 at $\{s=-\frac{1}{x_1}\}$ and that of F_2 at $\{s=-\frac{2}{x}+\frac{1}{x_1}\}$. If we write

$$F_1(s,x,y) \sim \sum_{k=1}^{\infty} F_{1,k}(x,y)\left(s+\frac{1}{x_1}\right)$$
$$F_2(s,x,y) \sim \sum_{k=1}^{\infty} F_{2,k}(x,y)\left(-s+\frac{2}{x}-\frac{1}{x_1}\right),$$

we find that the coefficients $F_{1,k}$ and $F_{2,k}$, satisfy a series of transport equations. Since the surface $\{s=-\frac{2}{x}+\frac{1}{x_1}\}$ does not intersect the boundary $\{x=0\}$, we are only concerned with the singularity along $\{s=-\frac{1}{x_1}\}$. The transport equation for $F_{1,1}$ is

$$\left(2\frac{\partial}{\partial x}+A\right)F_{1,1}(x,y)=0$$
$$F_{1,1}(x_1,y)=\frac{1}{2}f(x_1,y).$$

We have proved

Proposition 4.1. *Let $F\in W$, defined in (4.1), and let $f\in C_0^\infty(\overset{\circ}{X})$ be such that $\mathcal{R}_{1,-}(0,f)=F$. Then, for $x_1\in(0,\epsilon_0)$, $\mathcal{R}_{1,+}\mathcal{R}_{1,-}^{-1}\left(\mathcal{P}_{x_1}^- F\right)(s,y)$ has an expansion*

$$\mathcal{R}_{1,+}\mathcal{R}_{1,-}^{-1}\left(\mathcal{P}_{x_1}^- F\right)(s,y) = \frac{1}{2}f(x_1,y)\frac{|h_1|^{\frac{1}{4}}(x_1,y)}{|h_1|^{\frac{1}{4}}(0,y)}(s-\frac{1}{x_1})_+^0 + \text{smoother terms.} \tag{4.3}$$

Notice that the left hand side of (4.3) is determined by the scattering matrix, while the right hand side gives

$$\frac{1}{2}R_{1,-}^{-1}F(x_1,y)\frac{|h_1|^{\frac{1}{4}}(x_1,y)}{|h_1|^{\frac{1}{4}}(0,y)}(s-\frac{1}{x_1})_+^0 + \text{ smoother terms}$$

As a consequence of that we have that

$$R_{1,-}^{-1}F(x_1,y)\frac{|h_1|^{\frac{1}{4}}(x_1,y)}{|h_1|^{\frac{1}{4}}(0,y)}$$

is determined by the scattering matrix for every $F\in W$. That is if (X_1,g_1) and (X_2,g_2) satisfy the hypotheses of Theorem 1.3 then

$$\frac{|h_1|^{\frac{1}{4}}(x_1,y)}{|h_1|^{\frac{1}{4}}(0,y)}\mathcal{R}_{1,-}^{-1}F(x_1,y) = \frac{|h_2|^{\frac{1}{4}}(x_1,y)}{|h_2|^{\frac{1}{4}}(0,y)}\mathcal{R}_{2,-}^{-1}F(x_1,y), \tag{4.4}$$

$$\text{for every} \quad F\in W. \tag{4.5}$$

PROPOSITION 4.2. *If (X_1, g_1) and (X_2, g_2) satisfy the hypotheses of Theorem 1.3, there exists $\epsilon_0 > 0$ such that in a product decomposition $X \sim [0, \epsilon_0) \times \partial X$, where (1.3) holds, the coefficients $h_{1,ij}(x, y)$ and $h_{2,ij}$ of the tensors h_1 and h_2 satisfy $h_{1,ij}(x, y) = h_{2,ij}$.*

PROOF. First notice that if F and f are as in Proposition 4.1,

$$\mathcal{R}_{1,-}^{-1}\left(\frac{\partial^2}{\partial s^2}F\right) = (0, \Delta_{g_1} f).$$

So it follows from (4.3) that, modulo smoother terms,

$$\mathcal{R}_{1,+}\mathcal{R}_{1,-}^{-1}\left(\mathcal{P}_{x_1}^{-}\frac{\partial^2}{\partial s^2}F\right)(s, y) \equiv \frac{|h_1|^{\frac{1}{4}}(x_1, y)}{|h_1|^{\frac{1}{4}}(0, y)}\Delta_{g_1}\mathcal{R}_{1,-}^{-1}F(x_1, y)(s - \frac{1}{x_1})_+^0. \tag{4.6}$$

Doing the same for the metric g_2 we have, modulo smoother terms,

$$\mathcal{R}_{2,+}\mathcal{R}_{2,-}^{-1}\left(\mathcal{P}_{x_1}^{-}\frac{\partial^2}{\partial s^2}F\right)(s, y) \equiv \tag{4.7}$$

$$\frac{|h_2|^{\frac{1}{4}}(x_1, y)}{|h_2|^{\frac{1}{4}}(0, y)}\Delta_{g_2}\mathcal{R}_{2,-}^{-1}F(x_1, y)(x_1, y)(s - \frac{1}{x_1})_+^0 \tag{4.8}$$

Since the left hand sides of (4.6) and (4.8) are determined by the scattering matrix, it follows that

$$\frac{|h_2|^{\frac{1}{4}}(x_1, y)}{|h_2|^{\frac{1}{4}}(0, y)}\Delta_{g_2}\mathcal{R}_{2,-}^{-1}F(x_1, y)(x_1, y) = \frac{|h_1|^{\frac{1}{4}}(x_1, y)}{|h_1|^{\frac{1}{4}}(0, y)}\Delta_{g_1}\mathcal{R}_{1,-}^{-1}F(x_1, y).$$

Then using (4.5) and that $|h_1|(0, y) = |h_2|(0, y)$ we arrive at

$$|h_2|^{\frac{1}{4}}(x_1, y)\Delta_{g_2}\frac{|h_1|^{\frac{1}{4}}(x_1, y)}{|h_2|^{\frac{1}{4}}(x_1, y)}\mathcal{R}_{1,-}^{-1}F(x_1, y) = |h_1|^{\frac{1}{4}}(x_1, y)\Delta_{g_1}\mathcal{R}_{1,-}^{-1}F(x_1, y),$$
$$\text{for every} \quad F \in W.$$

Since F and x_1 are arbitrary, it follows from the definition of W that

$$|h_2|^{\frac{1}{4}}\Delta_{g_2}\frac{|h_1|^{\frac{1}{4}}}{|h_2|^{\frac{1}{4}}}f = |h_1|^{\frac{1}{4}}\Delta_{g_1}f, \quad \text{for every} \quad f \in C_0^\infty(\mathring{X}),$$
$$\text{with} \quad \operatorname{supp}(f) \subset \{x < \epsilon_0\}.$$

Therefore the differential operators $|h_2|^{\frac{1}{4}}\Delta_{g_2}\frac{|h_1|^{\frac{1}{4}}}{|h_2|^{\frac{1}{4}}}$ and $|h_1|^{\frac{1}{4}}\Delta_{g_1}$ are equal in $\{x < \epsilon_0\}$. But their principal parts are respectively

$$|h_1|^{\frac{1}{4}}\left(-x^4\frac{\partial^2}{\partial x^2} - x^2 h_2^{ij}\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\right)$$
$$\text{and} \quad |h_1|^{\frac{1}{4}}\left(-x^4\frac{\partial^2}{\partial x^2} - x^2 h_1^{ij}\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\right).$$

This implies that $h_1^{ij} = h_2^{ij}$. □

This determines the metric in a collar neighborhood of the boundary. One can then use the method of Belishev and Belishev-Kurylev [**1, 2**], see also [**9**] to show that this is the case in the whole manifold X. This is carried out in the asymptotically hyperbolic case in [**16**], and is identical in this case.

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