Semismall perturbations, semi-intrinsic ultracontractivity, and integral representations of solutions for parabolic equations

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Abstract:

In joint work with M. Murata. We study the connections between several contractivity properties of the semigroup associated to a second order elliptic operator L on a domain $D \subset \mathbb{R}^d$. Following Davies-Simon we say that D is intrinsic ultracontractive [**IU**], if for any $t > 0$, there exists a constant $C_t > 0$ such that

$$p(x, y, t) \leq C_t \; \phi_0(x)\phi_0(y), \quad x, y \in D,$$

where ϕ_0 is the normalized positive eigenfunction for the smallest eigenvalue λ_0. This property is related to the following perturbation codition.

[**SSP**] For $a < \lambda_0$. The constant function 1 is called a semismall perturbation of $L - a$ on D. If for any $\varepsilon > 0$ there exists a compact subset K of D such that

$$\int_{D \setminus K} G(x^0, z)G(z, y)dz \leq \varepsilon G(x^0, y), \qquad y \in D \setminus K,$$

where G is the Green function of $L - a$ on D, and x^0 is a fixed reference point in D. We will show that this condition implies the following weaker version of [**IU**].

[**SIU**] For any $t > 0$ and any compact subset K of D, there exist positive constants A and B such that

$$A \; \phi_0(x)\phi_0(y) \leq p(x, y, t) \leq B \; \phi_0(x)\phi_0(y), \quad x \in K, \quad y \in D.$$

Under the assumption [SSP] we establish an integral representation theorem of nonnegative solutions of the parabolic equation

$$(\partial_t + L)u = 0 \quad \text{in} \quad D \times I, \tag{0.1}$$

where $I = (0, T)$ with $0 < T \leq \infty$ or $I = (-\infty, 0)$.

In the case $I = (0, T)$, any nonnegative solution is represented uniquely by an integral on $(D \times \{0\}) \cup (\partial_M D \times [0, T))$, where $\partial_M D$ is the Martin boundary of D for the elliptic operator; and in the case $I = (-\infty, 0)$, any nonnegative solution is represented uniquely by the sum of an integral on $\partial_M D \times (-\infty, 0)$ and a constant multiple of a particular solution.